VOLUMES OF SOME SOLIDS WITH SMOOTH SURFACE AREAS

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In this paper, we applied an alternative method to calculate the volume of solids of revolution, if their surfaces are the smooth areas and these surfaces are described by parametric equations. We also present an example of calculation the volume of toroid, axiod, melonom, elliptic toroid, axoid (elliptical horn toroid), melonoid (elliptical spindle toroid), ellipsoid and elliptical rings with circular and elliptical cross-section.

K e y w o r d s: volume, solids of revolution, toroid, axiod, melonom, elliptic toroid, elliptical axoid, elliptical melonoid, ellipsoid, elliptical rings with circular and elliptical cross-section.

1 INTRODUCTION

In many physical problems, we encounter the necessity to determine the volume of the object (density, charge density, volume strain, etc.).The volume of solids in the Euclidean three-dimensional space is usually calculated using multiple integrals. Some simplification is achieved in the case of solids of revolution.

That problem of parametric descriptions of the surface areas of solids is investigated as a problem in the topological sense [3], [4]. The formula for the calculation of the volume of *n*-dimensional solids in the space E_n is proved in the papers [1] and [2] for the case that the surface areas are the smooth areas (respective smooth by parts) in a Euclidean space of the corresponding dimensions.

2 VOLUME OF SOLIDS OF REVOLUTION

Firstly, let us formulate the problem in general: Let

$X = [x_1, x_2, x_3]$ be points and its Cartesian co-ordinates in E_3,

$U = [u_1, u_2]$ the Cartesian coordinates of a point $U \in E_2$,

Ω the bounded closed domain in E_2,

$x_i(u_1, u_2), i = 1,2,3,$ given functions defined on some domain $O \subset E_2, \Omega \subset O$.

Let us also suppose that

- the vector $\mathbf{x}(\mathbf{u})$ function has almost everywhere in Ω the continuous partial derivatives $B_j^i = \frac{\partial x_i}{\partial u_j}$ for $i = 1,2,3,\ j = 1,2$;
- the rank of the matrix $\left(B_j^i\right)_{3x2}$ is equal to 2 almost everywhere in Ω;
- the subset $P^0 = \{\mathbf{x} \in E_3 | \mathbf{x} = \mathbf{x}(\mathbf{u}), \mathbf{u} \in \operatorname{int}\Omega\}$ of the set $P = \{\mathbf{x} \in E_3 | \mathbf{x} = \mathbf{x}(\mathbf{u}), \mathbf{u} \in \operatorname{int}\Omega\}$ is homeomorphic range of the set $\operatorname{int}\Omega$ in E_3.

Then, we can consider the closure W of the set P, which is called the 3-dimensional solid in the space E_3, is the boundary of it. The volume μW of the 3-dimensional solid W can be calculated by the formula, [1] and [2]

$$\mu W = V = \frac{1}{3}\iint_{\Omega} |\Delta(\mathbf{u})| du_1\, du_2 \tag{1}$$

where

$$\Delta(\mathbf{u}) = \begin{vmatrix} x_1(\mathbf{u}) & x_2(\mathbf{u}) & x_3(\mathbf{u}) \\ B_1^1 & B_1^2 & B_1^3 \\ B_2^1 & B_2^2 & B_2^3 \end{vmatrix}. \tag{2}$$

In the case of solids of revolution, we can use the following parameterization ($u_1 = u$, $u_2 = v$)

$$\begin{aligned} x_1 &= f(u)\cos v \\ x_2 &= f(u)\sin v \qquad v \in \langle 0,2\pi \rangle, u \in \langle a,b \rangle \\ x_3 &= g(u). \end{aligned} \tag{3}$$

Then

$$\Delta(u) = \begin{vmatrix} f(u)\cos v & f(u)\sin v & g(u) \\ f'(u)\cos v & f'(u)\sin v & g'(u) \\ -f(u)\sin v & f(u)\cos v & 0 \end{vmatrix} =$$

$$\begin{aligned} &= f(u)f'(u)g(u) - g'(u)f^2(u) = \\ &= f(u)[f'(u)g(u) - g'(u)f(u)] \end{aligned} \tag{4}$$

and the formula for calculating the volume is simplified into the relationship

$$V = \frac{1}{3}\int_0^{2\pi} dv \int_a^b |\Delta(u)|\, du = \frac{2\pi}{3}\int_a^b |\Delta(u)|\, du. \tag{5}$$

3 VOLUME OF TOROID

3.1 Volume of toroid

A torus is a surface of revolution generated by revolving a circle in E_3 about an axis that is coplanar with the circle (Fig. 1) and has no common point with it.

Using the symmetry of the toroid (Fig. 2), we can calculate the volume of only that part which lies above the plane (xy). Parametric equations of the semi-surface are

$$x_1 = f(u)\cos v = u\cos v$$
$$x_2 = f(u)\sin v = u\sin v \quad v \in \langle 0,2\pi\rangle, u \in \langle R-r, R+r\rangle$$
$$x_3 = g(u) = \sqrt{r^2-(u-R)^2} \tag{6}$$

where r denotes the radius of the rotating circle $k(S;r)$ and $R \rangle r$ is the distance of the centre $S=[R;0;0]$ from the z-axis of rotation.

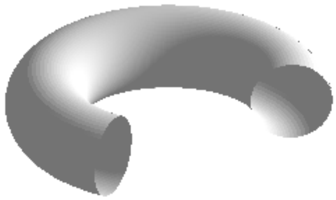

Figure 1: Part of torus

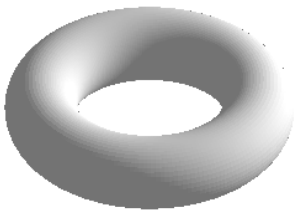

Figure 2: Toroid

In this particular case, the formula (4) takes the form

$$\Delta(u) = u\sqrt{r^2-(u-R)^2} + \frac{u^2(u-R)}{\sqrt{r^2-(u-R)^2}} \tag{7}$$

and the volume of a toroid is

$$V = 2\frac{2\pi}{3}\int_{r-R}^{r+R}|\Delta(u)|\,du =$$

$$= \frac{4\pi}{3}\left[\frac{3Rr^2}{2}\arcsin\frac{u-R}{r} - \left(\frac{R^2}{2}+r^2-\frac{Ru}{2}\right)\sqrt{r^2-(u-R)^2}\right]_{R-r}^{R+r} =$$

$$= 2\pi^2Rr^2 \tag{8}$$

3.2 Volume of axoid

An axoid (horn toroid) is generated by revolving a circle in E_3 about an axis that is a tangent to the circle, (Fig. 3). If the circle rotates about the z-axis then the parametric equations of the semi-surface are

$$x_1 = f(u)\cos v = u\cos v$$
$$x_2 = f(u)\sin v = u\sin v \qquad v \in \langle 0,2\pi\rangle, u \in \langle 0,2r\rangle$$
$$x_3 = g(u) = \sqrt{r^2-(u-r)^2} \tag{9}$$

Then

$$\Delta(u) = u\sqrt{r^2-(u-r)^2} + \frac{u^2(u-r)}{\sqrt{r^2-(u-r)^2}} \tag{10}$$

and the volume of an axoid (Fig. 4) is

$$V = 2\frac{2\pi}{3}\int_0^{2r}|\Delta(u)|\,du =$$

$$= \frac{4\pi}{3}\left[\frac{3r^3}{2}\arcsin\frac{u-r}{r} - \left(\frac{3r^2}{2}-\frac{ru}{2}\right)\sqrt{r^2-(u-r)^2}\right]_0^{2r} =$$

$$= 2\pi^2r^3 \tag{11}$$

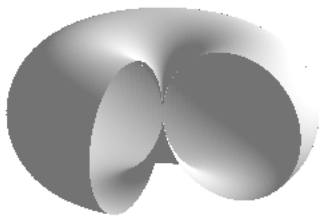

Figure 3: Part of axoid surface (horn torus)

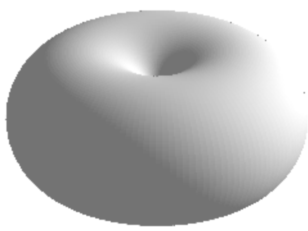

Figure 4: Axoid (horn toroid)

3.3 Volume of melonoid

A melonoid (Fig. 5) is a surface in E_3 generated by revolving a circle about an axis when this axis is a chord of the circle.

Using the same parametric equations as in the previous cases, the calculation of the volume of a melonoid (Fig. 6) must be carried out in two steps:

a) volume V_0 determined by the "outer" surface,

$$V_0 = 2\frac{2\pi}{3}\int_0^{2r}|\Delta(u)|\,du =$$

$$= \frac{4\pi}{3}\left|\left[\frac{3}{2}Rr^2\arcsin\frac{u-R}{r} - \left(\frac{R^2}{2}+r^2-\frac{Ru}{2}\right)\sqrt{r^2-(u-R)^2}\right]_0^{R+r}\right| =$$

$$= \frac{4\pi}{3}\left\|\left[\frac{3}{2}Rr^2\left(\frac{\pi}{2}+\arcsin\frac{R}{r}\right)-\left(\frac{R^2}{2}+r^2\right)\sqrt{r^2-R^2}\right]\right\| \quad (12)$$

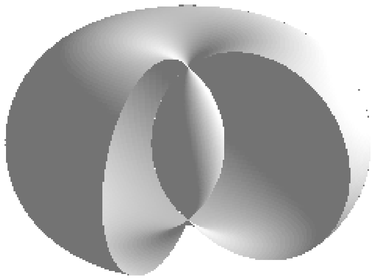

Figure 5: Part of melonoid surface (spindle torus)

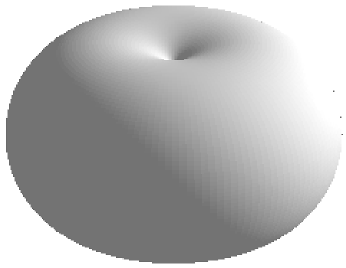

Figure 6: Melonoid (Spindle toroid)

b) volume V_1 determinated by the inner "surface"

$$V_1 = 2\frac{2\pi}{3}\int_{R-r}^{0}\left|\Delta(u)\right| du =$$

$$= \frac{4\pi}{3}\left\|\left[\frac{3}{2}Rr^2\arcsin\frac{u-R}{r}-\left(\frac{R^2}{2}+r^2-\frac{Ru}{2}\right)\sqrt{r^2-(u-R)^2}\right]_{R-r}^{0}\right\| =$$

$$= \frac{4\pi}{3}\left\|\left[\frac{3}{2}Rr^2\left(\frac{\pi}{2}-\arcsin\frac{R}{r}\right)-\left(\frac{R^2}{2}+r^2\right)\sqrt{r^2-R^2}\right]\right\| \quad (13)$$

The volume of the "hollow" body is the difference between these two volumes, i.e.

$$V = V_0 - V_1 = 4\pi Rr^2 \arcsin\frac{R}{r} \quad (14)$$

3.4 Volume of sphere

A sphere can be considered as a degenerate case of a toroid when the axis of rotation is a diameter of the rotating circle. That is why its volume is given by formula (12) for $R = 0$, i.e.

$$V = \frac{4}{3}\pi r^3 \quad (15)$$

4 VOLUME OF ELLIPTICAL TOROID

An elliptical torus is a surface of revolution generated by revolving an ellipse in E_3 about an axis that is coplanar with the circle (Fig. 7) and has no common point with it.

Using the symmetry of the elliptical toroid (Fig. 8), we can calculate the volume of only that part which lies above the plane (xy). Parametric equations of the semi-surface are

$$x_1 = f(u)\cos v = u\cos v$$
$$x_2 = f(u)\sin v = u\sin v \quad v \in \langle 0, 2\pi\rangle, u \in \langle R-a, R+a\rangle$$
$$x_3 = g(u) = \frac{b}{a}\sqrt{r^2-(u-R)^2} \quad (16)$$

where a, b denote the semi-axes of the rotating ellipse $e = (S; a, b)$ and $R \rangle a$ is the distance of the centre $S = [R,0,0]$ from the z-axis of rotation.

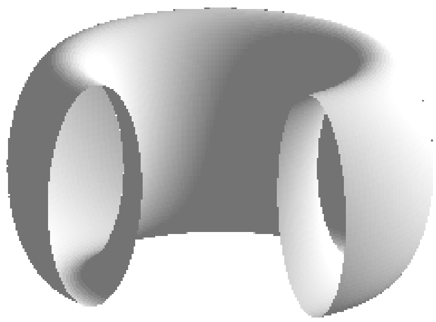

Figure 7: Part of elliptical torus

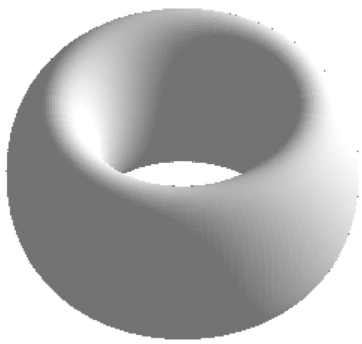

Figure 8: Elliptical toroid

In this particular case, the formula (4) takes the form

$$\Delta(u) = \frac{b}{a}\left(u\sqrt{r^2-(u-R)^2}+\frac{u^2(u-R)}{\sqrt{r^2-(u-R)^2}}\right) \quad (17)$$

and the volume of an elliptical toroid is

$$V = 2\frac{2\pi}{3}\frac{b}{a}\int_{R-a}^{R+a}\left|\Delta(u)\right| du =$$

$$= \frac{4\pi b}{3a}\left|\left[\frac{3}{2}Ra^3 \arcsin\frac{u-R}{a} - \left(\frac{R^2}{2} + a^2 - \frac{Ru}{2}\right)\sqrt{a^2-(u-R)^2}\right]_{R-a}^{R+a}\right| =$$

$$= 2\pi^2 Ra^2 b \tag{18}$$

4.1 Volume of elliptical axoid

An elliptical axoid (elliptical horn torus) is generated by revolving an ellipse in E_3 about an axis that is a tangent to the circle, (Fig. 9). If the circle rotates about the z-axis then the parametric equations of the semi-surface are

$$\begin{aligned} x_1 &= f(u)\cos v = u\cos v \\ x_2 &= f(u)\sin v = u\sin v \qquad v \in \langle 0,2\pi\rangle, u \in \langle 0,2r\rangle \\ x_3 &= g(u) = \frac{b}{a}\sqrt{a^2-(u-a)^2} \end{aligned} \tag{19}$$

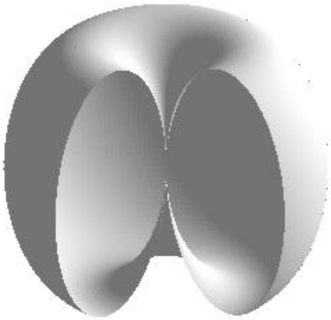

Figure 9: Part of elliptical axoid surface

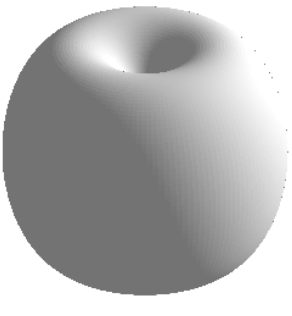

Figure 10: Elliptical axoid

Then

$$\Delta(u) = \frac{b}{a}\left(u\sqrt{a^2-(u-a)^2} + \frac{u^2(u-a)}{\sqrt{a^2-(u-a)^2}}\right) \tag{20}$$

and the volume of an elliptical axoid (Fig. 10) is

$$V = 2\frac{2\pi}{3}\frac{b}{a}\int_{0}^{2a}\left|\Delta(u)\right| du =$$

$$= \frac{4\pi b}{3a}\left|\left[\frac{3}{2}a^3 \arcsin\frac{u-a}{a} - \left(\frac{3a^2}{2} + a^2 - \frac{au}{2}\right)\sqrt{a^2-(u-a)^2}\right]_{0}^{2a}\right| =$$

$$= 2\pi^2 a^2 b \tag{21}$$

4.2 Volume of elliptical melonoid

An elliptical melonoid (Fig. 11) is a surface in E_3 generated by revolving an ellipse about an axis when this axis is a chord of the ellipse.

Using the same parametric equations as in previous cases, the calculation of the volume of a melonoid (Fig. 12) must be carried out in two steps:

a) volume V_0 determined by the "outer" surface,

$$V_0 = 2\frac{2\pi}{3}\frac{b}{a}\int_{0}^{a+R}\left|\Delta(u)\right| du =$$

$$= \frac{4\pi b}{3a}\left|\left[\frac{3}{2}Ra^2 \arcsin\frac{u-R}{a} - \left(\frac{R^2}{2} + a^2 - \frac{Ru}{2}\right)\sqrt{a^2-(u-R)^2}\right]_{0}^{R+a}\right| =$$

$$= \frac{4\pi b}{3a}\left|\left[\frac{3}{2}Ra^2\left(\frac{\pi}{2} + \arcsin\frac{R}{a}\right) - \left(\frac{R^2}{2} + a^2\right)\sqrt{a^2-R^2}\right]\right| \tag{22}$$

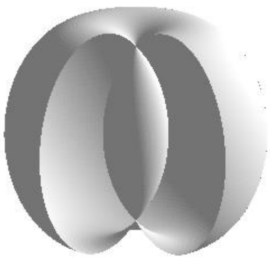

Figure 11: Part of elliptical melonoid surface

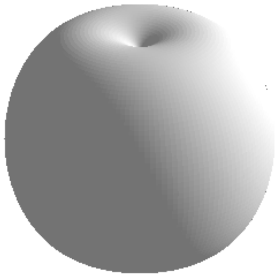

Figure 12: Elliptical melonoid

b) volume V_1 is determined by the inner "surface"

$$V_1 = 2\frac{2\pi}{3}\frac{b}{a}\int_{R-a}^{0}|\Delta(u)|\,du =$$

$$= \frac{4\pi b}{3a}\left|\left[\frac{3}{2}Ra^2\arcsin\frac{u-R}{a} - \left(\frac{R^2}{2}+a^2-\frac{Ru}{2}\right)\sqrt{a^2-(u-R)^2}\right]_{R-a}^{0}\right| =$$

$$= \frac{4\pi b}{3a}\left|\left[\frac{3}{2}Ra^2\left(\frac{\pi}{2}-\arcsin\frac{R}{a}\right) - \left(\frac{R^2}{2}+a^2\right)\sqrt{a^2-R^2}\right]\right| \tag{23}$$

The volume of the "hollow" body is the difference between these two volumes, i.e.

$$V = V_0 - V_1 = 4\pi Rab\arcsin\frac{R}{a} \tag{24}$$

4.3 Volume of an ellipsoid

An ellipsoid can be considered as a degenerate case of an elliptical toroid when the axis of rotation is a minor semi-axis of the rotating ellipse. That is why its volume is given by formula (22) for $R = 0$, i.e.

$$V = \frac{4}{3}\pi a^2 b \tag{25}$$

5 VOLUME OF ELLIPTICAL RINGS

5.1 Volume of elliptical ring with elliptical cross-section

An elliptical ring with an elliptical cross-section is a surface formed by moving an ellipse $e(S,c,d)$ in an elliptical orbit $\overline{e}(O,a,b)$ that is perpendicular to the plane of the ellipse *e* (Fig. 13 and Fig.14).

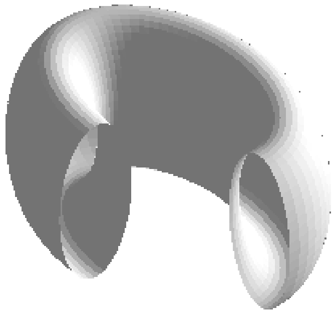

Figure 13: Part of elliptical ring surface with elliptical cross-section

Figure 14: Elliptical ring with elliptical cross-section

Parametric equations of the surface are

$$\begin{aligned} x_1 &= (a + c\cos v)\cos u \\ x_2 &= (b + c\cos v)\sin u \\ x_3 &= d\sin v\,, \quad u \in \langle 0, 2\pi\rangle, \quad v \in \langle 0, 2\pi\rangle \end{aligned} \tag{26}$$

where a and b denote the semi-axes of the trajectory of the moving ellipse $e(S,c,d)$. We suppose that the surface is not intersecting, i.e. $\min(a,b) \ge c$.

In this particular case, the general formula (2) takes the form

$$\Delta(u,v) = \begin{vmatrix} (a+c\cos v)\cos u & (b+c\cos v)\sin u & d\sin v \\ -(a+c\cos v)\sin u & (b+c\cos v)\cos u & 0 \\ -c\sin v\cos u & -c\sin v\sin u & d\cos v \end{vmatrix} =$$

$$= abc\cos v + cd(a+b)\cos^2 v + acd\sin^2 u\sin^2 v +$$

$$+ c^2 d\sin^2 v\cos v + bcd\sin^2 v\cos^2 u \tag{27}$$

and the volume of the ring is equal to

$$V = \frac{1}{3}\int_0^{2\pi}\int_0^{2\pi}\left|\Delta(u)\right| dudv =$$

$$= \frac{1}{3}\int_0^{2\pi} [u\{abc\cos v + cd\left(a+b\right)\cos^2 v + c^2 d\sin^2 v\cos v\} +$$

$$+ \frac{1}{2} cd\sin^2 v\{a(u - \frac{\sin 2u}{2}) + b(u + \frac{\sin 2u}{2})\}]_0^{2\pi} dv =$$

$$= \frac{1}{3}\int_0^{2\pi} [2\pi\{abc\cos v + cd\left(a+b\right)\cos^2 v + c^2 d\sin^2 v\cos v\} +$$

$$+ \pi cd\sin^2 v(a+b)]dv =$$

$$= \frac{1}{3}[2\pi\{abc\sin v + cd\left(a+b\right)\frac{1}{2}(v + \frac{\sin 2v}{2}) + \frac{1}{3}c^2 d\sin^3 v\} +$$

$$+ \frac{1}{2}\pi cd(a+b)(v - \frac{\sin 2v}{2})]_0^{2\pi} = \left(a+b\right)cd\pi^2 \qquad (28)$$

5.2 Volume of elliptical ring with circular cross-sections

An elliptical ring with a circular cross-section is a surface formed by moving a circle $k(S;r)$ in an elliptical orbit that is perpendicular to the circle k (Fig. 15 and 16).

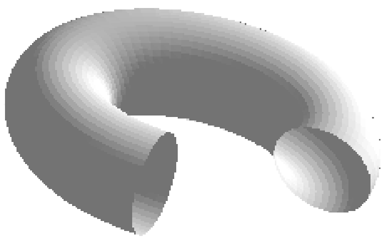

Figure 15: Part of elliptical ring surface with circular cross-section

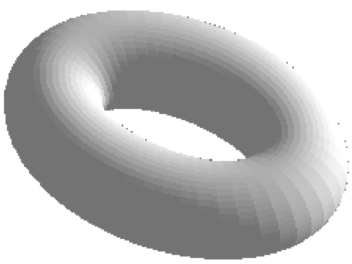

Figure 16: Elliptical ring with circular cross-section

As a circle $k(S;r)$ can be taken as a special case of an ellipse if $c = d = r$, we can use the previous results, i.e.

$$\Delta(u,v) = \begin{vmatrix} (a + r\cos v)\cos u & (b + r\cos v)\sin u & r\sin v \\ -(a + r\cos v)\sin u & (b + r\cos v)\cos u & 0 \\ -r\sin v\cos u & -r\sin v\sin u & r\cos v \end{vmatrix} =$$

$$= abr\cos v + r^2\left(a+b\right)\cos^2 v + ar^2\sin^2 u\sin^2 v +$$

$$+ r^3\sin^2 v\cos v + br^2\sin^2 v\cos^2 u \qquad (29)$$

and the volume is equal to

$$V = \frac{1}{3}\int_0^{2\pi}\int_0^{2\pi}\left|\Delta(u)\right| dudv = \left(a+b\right)r^2\pi^2. \qquad (30)$$

5 CONCLUSION

General alternative method of calculating the volume of solids is dependent on finding a suitable parameterization of their surfaces. In the case of solids of revolution, the problem is somewhat easier, because it is necessary only to find a parameterization of the curve, which is the plane section of the solid by a plane containing the axis of rotation.

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