LINEAR THREE – HELICAL SURFACES

Tatiana Olejníková

The paper deals with analytical representation of the linear three – helical surface, which is created by three – helical movement of the straight line. Three–helical movement is compound from three helical movements, while axis of the third helical movement screwes about axis of the second and this screwes about the first one simultaneously. Classification of these surfaces and some of their geometric properties are described. Ssurfaces are displayed in programme MAPLE.

K e y w o r d s. three – helical movement, compound helix, linear three – helical surface

1 INTRODUCTION

Three – helical movement is compound from three helical movements determined by the three axes ${}^1o, {}^2o, {}^3o$. The compound helix is created by the three – helical movement of a point, which screws about the axis ${}^3o$ with angular velocity w_3, pitcht of the helix ${}^3v$ and orientation determined by the parameter $q_3 = \pm 1$, the axis ${}^3o$ screws simultaneously about the axis ${}^2o$ with angular velocity w_2, pitcht of the helix ${}^2v$ and orientation determined by the parameter $q_2 = \pm 1$ and the axis ${}^2o$ screws simultaneously about the axis ${}^1o$ with angular velocity w_1, pitcht of the helix ${}^1v$ and orientation determined by the parameter $q_1 = \pm 1$, where $q_i = +1$ holds for right-handed and $q_i = -1$ for the left-handed screwing, (i=1, 2, 3). Linear three–helical surface is created by the three – helical movement of the line *p*, where every point of this line creates one compound helix.

2 ANALYTICAL REPRESENTATION OF LINEAR THREE – HELICAL SURFACE

Let us define the three –helical movement as a compound movement from three helical movements, which are determined by the axes ${}^1o, {}^2o, {}^3o$. Let us analytically describe this movement for case the axes of these movements are in the relative position: ${}^1o = z$, ${}^2o \parallel {}^1o$, ${}^3o \not\parallel {}^2o$ and the point $P = (x_0, y_0, z_0, 1)$ is located on the line $p \parallel {}^3o$. The axis ${}^3o$ locates in the plane parallel to the coordinate plane *xz* in the distance d_2, and it is determined by the points $A(0, d_2, 0)$, $B(d_3, d_2, d_4)$, where α is the angle between the axis ${}^3o$ and plane *xy* (Fig. 1).

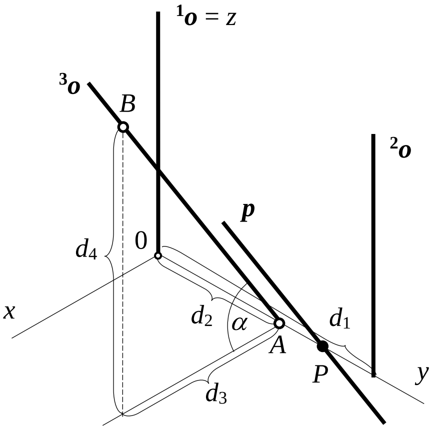


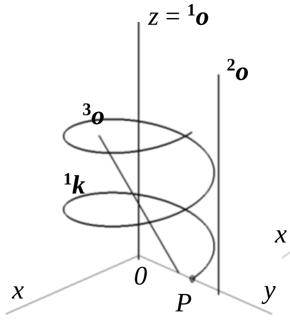
Figure 1

The transformation matrix of the helical movement with the axis ${}^1o$, which is determined by the angular velocity $w_1(v) = m_1 v$, orientation determined by the parameter $q_1 = \pm 1$ and reduced pitch of the helix ${}^1v_0 = {}^1v / 2\pi$ is for $v \in \langle 0, 2\pi \rangle$

$$\mathbf{T}_1(v) = \begin{pmatrix} \cos w_1(v) & q_1 \sin w_1(v) & 0 & 0 \\ -q_1 \sin w_1(v) & \cos w_1(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & {}^1v_0\, w_1(v) & 1 \end{pmatrix} \quad (1)$$

By the equation (2) it is expressed the vector function of the helix ${}^{1}k$ created by the screwing of the point *P* about the axis ${}^{1}o$, which is displayed in Fig. 2

$$\mathbf{r}_{k1}(v)=(x_{k1},y_{k1},z_{k1},1)=(x_0,y_0,z_0,1).\mathbf{T}_1(v) \quad (2)$$



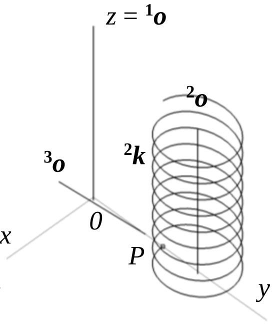


Figure 2 Figure 3

The transformation matrix of the helical movement with the axis ${}^{2}o$, which is determined by angular velocity $w_2(v)=m_2 w_1(v)$, orientation determined by the parameter $q_2=\pm 1$ and reduced pitch of the helix ${}^{2}v_0={}^{2}v/2\pi$ is for $v\in\langle 0,2\pi\rangle$

$$\mathbf{T}_2(v)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&-d_1&0&1\end{pmatrix}.$$

$$.\begin{pmatrix}\cos w_2(v)&q_2\sin w_2(v)&0&0\\-q_2\sin w_2(v)&\cos w_2(v)&0&0\\0&0&1&0\\0&d_1&{}^{2}v_0\,w_2(v)&1\end{pmatrix} \quad (3)$$

By the equation (4) it is expressed the vector function of the helix ${}^{2}k$ created by the screwing of the point *P* about the axis ${}^{2}o$, which is displayed in Fig. 3

$$\mathbf{r}_{k2}(v)=(x_0,y_0,z_0,1).\mathbf{T}_2(v) \quad (4)$$

The transformation matrix of the helical movement with the axis ${}^{3}o$, which is determined by angular velocity

$$w_3(v)=m_3 w_2(v)=m_3\, m_2\, m_1\, w_1(v),$$

orientation determined by the parameter $q_3=\pm 1$ and reduced pitch of the helix ${}^{3}v_0={}^{3}v/2\pi$ is for $v\in\langle 0,2\pi\rangle$ product of the three matrices

$$\mathbf{T}_2(v)=\mathbf{M_1}\,.\,\mathbf{M_2}\,.\,\mathbf{M_3}, \quad (5)$$

where

$$\mathbf{M_1}=\begin{pmatrix}\cos\alpha&0&-\sin\alpha&0\\0&1&0&0\\\sin\alpha&0&\cos\alpha&0\\0&-d_2&0&1\end{pmatrix},$$

$$\mathbf{M_2}=\begin{pmatrix}1&0&0&0\\0&\cos w_3(v)&q_3\sin w_3(v)&0\\0&-q_3\sin w_3(v)&\cos w_3(v)&0\\0&0&0&1\end{pmatrix},$$

$$\mathbf{M_3}=\begin{pmatrix}\cos\alpha&0&\sin\alpha&0\\0&1&0&0\\-\sin\alpha&0&\cos\alpha&0\\0&d_2&0&1\end{pmatrix}.$$

By the equation (6) it is expressed the vector function of the helix ${}^{3}k$ created by the screwing of the point *P* about the axis ${}^{3}o$, which is displayed in Fig. 4

$$\mathbf{r}_{k3}(v)=(x_0,y_0,z_0,1).\mathbf{T}_3(v). \quad (6)$$

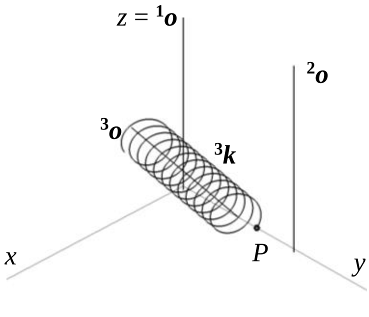


Figure 4

The axis ${}^{2}o$ creates the linear helical surface ${}^{1}\mathbf{\Omega}$ by its helical movement about the axis ${}^{1}o$ determined by the transformation matrix (1). This linear helical surface is cylindrical with vector function expressed in (7) (Fig. 5)

$$\mathbf{P}_{\mathrm{p}1}(u,v) = (0,\ d_1, u, 1).\mathbf{T}_1(v),\quad u \in \langle 0,1\rangle,\ v \in \langle 0, 2\pi\rangle. \tag{7}$$

The axis ${}^{3}o$ creates the linear helical surface by its helical movement about the axis ${}^{2}o$ determined by the transformation matrix (3). This linear helical surface is opened clinogonal with vector function expressed in (8) (Fig. 6)

$$\mathbf{P}_{\mathrm{p}2}(u,v) = (d_3 u,\ d_2, d_4 u, 1).\mathbf{T}_2(v),\quad u \in \langle 0,1\rangle,\ v \in \langle 0, 2\pi\rangle. \tag{8}$$

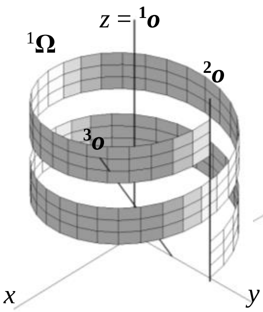


Figure 5

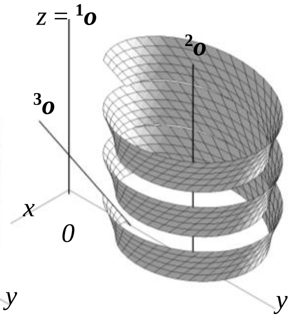


Figure 6

The axis ${}^{3}o$ creates the compound linear helical surface ${}^{2}\mathbf{\Omega}$ with vector function (9) (Fig.7) by its helical movement about the axis ${}^{2}o$, which screws about the axis ${}^{1}o$ simultaneously. This surface has two branches on one screw because angular velocity of the helical movement with the axis ${}^{3}o$ is double to the angular velocity of helical movement with the axis ${}^{2}o$,

$$\mathbf{P}_{\mathrm{p}3}(u,v) = (d_3 u,\ d_2, d_4 u, 1)\,.\,\mathbf{T}_2(v)\,.\,\mathbf{T}_1(v),\quad u \in \langle 0,1\rangle,\ v \in \langle 0, 2\pi\rangle. \tag{9}$$

In Fig. 8 there are displayed the helical surfaces ${}^{1}\mathbf{\Omega}$ a ${}^{2}\mathbf{\Omega}$ created by helical movement of the axis ${}^{2}o$ and by compound helical movement of the axis ${}^{3}o$ about the axis ${}^{1}o$ together.

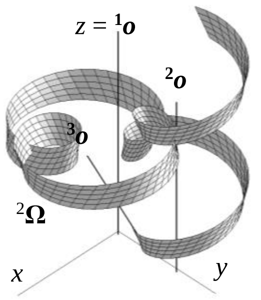


Figure 7

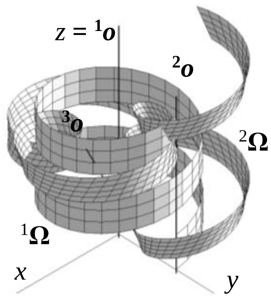


Figure 8

Point $P = (x_0, y_0, z_0, 1)$ moving by the helical movement about the axis ${}^{3}o$ with angular velocity $w_3(v) = m_3 w_2(v) = m_3\, m_2\, m_1\, w_1(v)$, orientation determined by the parameter $q_3 = \pm 1$, reduced pitch of the helix ${}^{0}v_3$ creates the curve *k*, whereas the axis ${}^{3}o$ screws about the axis ${}^{2}o$ with angular velocity $w_2(v) = m_2\, w_1(v)$, orientation determined by the parameter $q_2 = \pm 1$, reduced pitch of the helix ${}^{0}v_2$, and the axis ${}^{2}o$ screws about the axis ${}^{1}o$ with angular velocity $w_1(v) = m_1\, v$, orientation determined by the parameter $q_1 = \pm 1$, reduced pitch of the helix ${}^{0}v_1$ simultaneously. Curve *k* has $m_2 = 3$ branches on its one screw, and every branch has $m_3 = 6$ subbranches (Fig. 9). The vector function of the curve *k* is for $v \in \langle 0, 2\pi\rangle$ in (10).

$$\mathbf{r}(v) = (x_0, y_0, z_0, 1)\,.\,\mathbf{T}_3(v)\,.\,\mathbf{T}_2(v)\,.\,\mathbf{T}_1(v)\,. \tag{10}$$

Line *p* (passing thrue the point *P*) determined by the vector function $\mathbf{r}(x_1(u), y_1(u), z_1(u), 1), u \in \langle 0,1\rangle$ creates the linear three – helical surface ${}^{3}\mathbf{\Omega}$ by its screwing about the axis ${}^{3}o$ with angular velocity $w_3(v) = m_3 w_2(v) = m_3\, m_2\, m_1\, w_1(v)$, orientation determined by the parameter $q_3 = \pm 1$, reduced pitch of the helix ${}^{0}v_3$, whereas the axis ${}^{3}o$ screws about the axis ${}^2o$ with angular velocity $w_2(v) = m_2\, w_1(v)$, orientation determined by the parameter $q_2 = \pm 1$, reduced pitch of the helix ${}^0v_2$, and the axis ${}^2o$ screws about the axis ${}^1o$ with angular velocity $w_1(v) = m_1\, v$, orientation determined by the parameter $q_1 = \pm 1$, reduced pitch of the helix ${}^0v_1$ simultaneously. This surface has $m_2 = 3$ branches on its one screw, and every branch has $m_3 = 6$ subbranches. The vector function of the surface ${}^3\mathbf{\Omega}$ is for $v \in \langle 0, 2\pi \rangle$ in (11).

$$\mathbf{P}(u,v) = \left(x_1(u), y_1(u), z_1(u), 1\right) . \mathbf{T}_3(v) . \mathbf{T}_2(v) . \mathbf{T}_1(v), \quad u \in \langle 0,1 \rangle, \quad v \in \langle 0, 2\pi \rangle \qquad (11)$$

In Fig. 10 there are displayed the surface ${}^2\mathbf{\Omega}$ and the curve k.

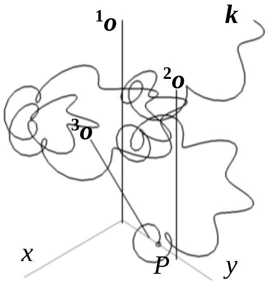


Figure 9

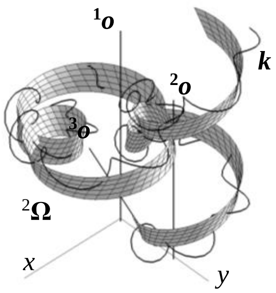


Figure 10

In Fig. 11 there are displayed the surface ${}^1\mathbf{\Omega}$ created by moving of the axis ${}^2o$, surface ${}^2\mathbf{\Omega}$ created by moving of the axis ${}^3o$ and the surface ${}^3\mathbf{\Omega}$ created by moving of the line p about the axes ${}^1o, {}^2o, {}^3o$ together. The linear surface ${}^3\mathbf{\Omega}$ is created by moving of the line p parallel to the axis ${}^3o$ determined by the parameters: $m_2 = 3, m_3 = 6, q_1 = q_2 = q_3 = +1$. This surface is in Fig. 12, and the case the line p is intersection to the axis ${}^3o$ is in Fig. 13 and if p is skew to the axis ${}^3o$ is in Fig. 14. This surfaces have the same parameters, the first is of cylindrical, the second of conical and the third is of hyperboloidal type.

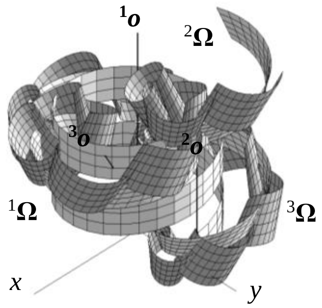


Figure 11

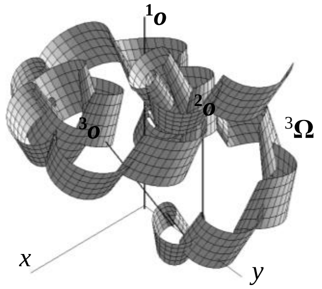


Figure 12

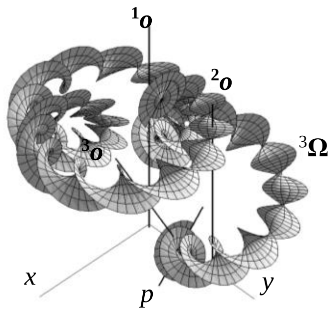


Figure 13

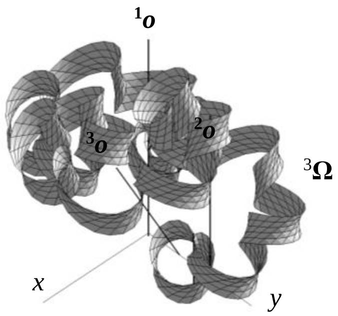


Figure 14

3 CLASSIFICATION OF THE LINEAR THREE – HELICAL SURFACES

The classification of the family of linear three – helical surfaces can be done according to the relative position of the axes ${}^1o, {}^2o, {}^3o$ and the straight line *p*.

The surfaces can be distributed into the three types **A**, **B**, **C** according to the relative position of the axes ${}^1o$ and ${}^2o$, into the three types **I**, **II**, **III** according to the relative position of the axes ${}^2o$ and ${}^3o$ and into the three types **1**, **2**, **3** according to the relative position of the line *p* and the axis ${}^3o$ (Tab. 1).

Tab. 1: Classification of the surfaces

Axes, line *p*	Types of the surfaces		
${}^2o, {}^1o$	**A**	**B**	**C**
	${}^2o \parallel {}^1o$	${}^2o \times {}^1o$	${}^2o \,/\, {}^1o$
${}^3o, {}^2o$	**I**	**II**	**III**
	${}^3o \parallel {}^2o$	${}^3o \times {}^2o$	${}^3o \,/\, {}^2o$
$p, {}^3o$	**1**	**2**	**3**
	$p \parallel {}^3o$	$p \times {}^3o$	$p \,/\, {}^3o$

4 DISPLAY OF THE LINEAR THREE – HELICAL SURFACES

In Figs. 15 – 19 there are displayed the some linear three – helical surfaces created by the three-helical movement of the line *p*. There are displayed also fronted possition of the axes ${}^1o, {}^2o, {}^3o$. These linear three–helical surfaces are of any types **A I 1**, A **I 2**, ... ,**C III 3** determined by the parameters $(m_1, m_2, m_3, q_1, q_2, q_3)$.

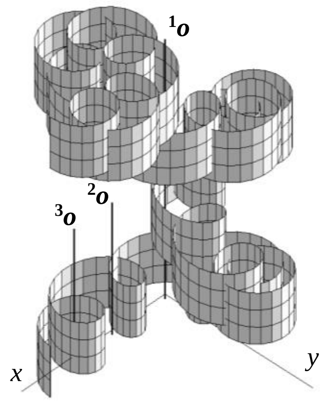


Figure 15: Type **A I 1**

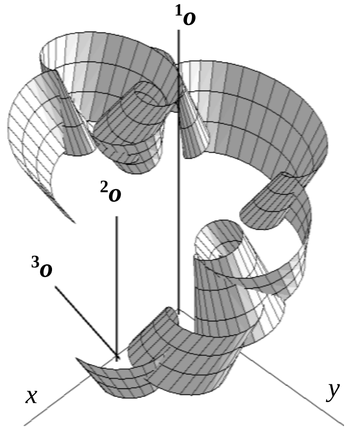


Figure 16: Type **A II 3**

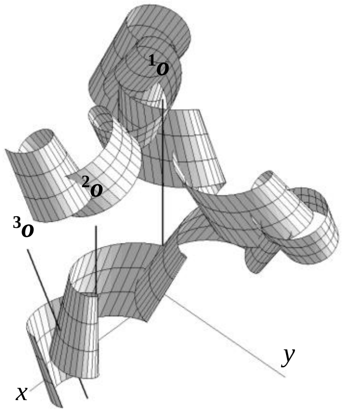


Figure 17: Type **A III 1**

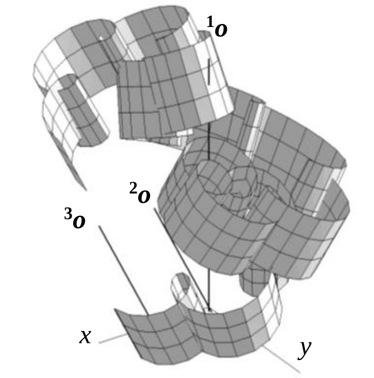


Figure 18: Type **B I 1**

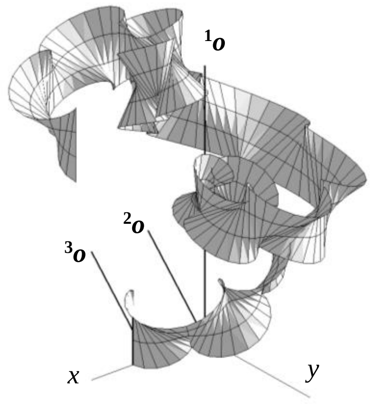


Figure 18: Type **B I 2**

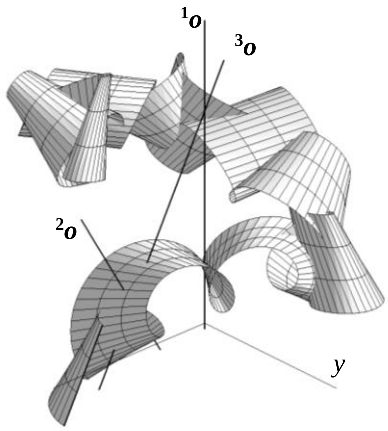


Figure 19: Type **B II 1**

5 CONCLUSION

As the conclusion, it can be summarized that the presented family of linear three – helical surfaces serves as an endlessly rich source of inspiration for artistic and design purposes. Their unusually complex forms are obtained in a relatively simple way of composite spatial transformations.

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AUTHOR ADDRESS

RNDr. Tatiana Olejníková, PhD.
Department of Applied Mathematics
Faculty of Civil Engineering TU Košice
Vysokoškolská 4, 042 00 Košice
E-mail: tatiana.olejnikova@tuke.sk