NON - SPHERICAL SURFACES, FOR WHICH TWO SYSTEMS OF LINES ARE CIRCLES

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The surfaces, for which two systems of lines are circles, are presented in this paper despite the fact that they are not a spherical surface or torus. The presented surfaces are views of linear surfaces in a spherical inversion, specifically the linear surfaces of rotation and the hyperbolic paraboloid. The first section of this paper deals with a circular inversion and its properties. The second one with a spherical inversion and in the third section of this paper, we devote to the visualization of before mentioned linear surfaces in a spherical inversion.

K e y w o r d s. circular inversion, spherical inversion, non-spherical surfaces

1 CIRCULAR INVERSION

Circular inversion is a special case of quadratic geometrical transformation. It is a nonlinear mapping in a plane [3].

Let a circle k with a centre S and a radius r, $k = (S,r)$ be in an Euclidean plane. The mapping in a plane that refers a point $X \neq S$ to the point X' lying on a half-line SX so, that $|SX|.\ |SX'| = r^2$ is called a circular inversion (CI), Fig.1.

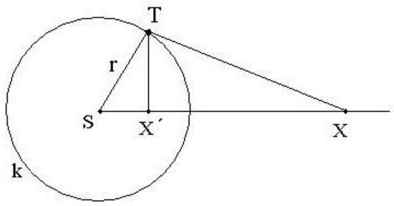

Fig. 1: The view of a point X in CI

Circle k is called a director or reference circle of CI (in some literature it is referred as the circle of inversion). The point S is the centre of CI and r^2 is the power of CI. The point S does not have a view in CI. We will define the view of the point S as an ideal point $\{\infty\}$, point at infinity of a plane. Euclidean plane which is expanded by one ideal point is called a Möbius plane.

Properties of circular inversion which result from its definition are:

- If $X \in k$, then $X=X'$. The director circle is a set of invariant points.
- If X is an interior of a director circle, then an inverse image X' is an exterior point of the circle k and vice versa.
- To the straight line passing through the centre of CI, corresponds the same straight line. It is then an invariant line, but not a point invariant line, p = p´ but X ≠ X´.
- To the straight line not passing through the centre of CI, corresponds a circle passing through the centre of CI.
- To the straight line, which is the tangent of the director circle, corresponds a circle passing through a common tangent point and a centre of a director circle and its radius is r/2.
- If the circle is not passing through the centre of CI, its inverse image is once again a circle.
- A circle, which passing through the director circle orthogonally, is invariant, but not a point invariant circle.
- CI is an involutory mapping, points X and X´ are inverse to each other, (X´)´=X.
- CI is a conformal mapping, which conserves angles. Two arbitrary straight lines in a plane are intersecting in the same angle as their inverse circles.

If coordinates of the points are $S(x_0,y_0)$, $X(x,y)$, $X'(x',y')$ and the radius of the director circle is r, then the coordinates of the point X' are:

$$x' = x_0 + \frac{r^2(x - x_0)}{(x - x_0)^2 + (y - y_0)^2},$$

$$y' = y_0 + \frac{r^2(y - y_0)}{(x - x_0)^2 + (y - y_0)^2}. \qquad (1)$$

Inverse image of a system of parallel straight lines is a parabolic set of circles, which are passing through the centre of the director circle, Fig.2.

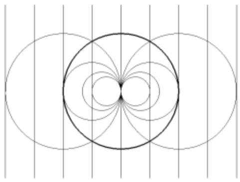

Fig. 2

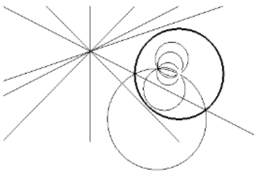

Fig. 3

A set of intersecting straight lines is mapping in CI into a hyperbolic set of circles, which are passing through two points, the centre of the director circle and the view of common point, Fig.3

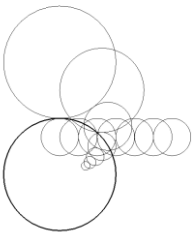

Fig. 4

On the figure 4, there is the inverse image of circles with the same radius, for which the centre lies on one common straight line.

2 SPHERICAL INVERSION

By the extension of CI from 2D into 3D, we get spherical inversion (SI). Instead of a director circle, we have a director spherical surface (DSS). Let its centre be a point $S(x_0,y_0,z_0)$ and a radius r. The inverse point $X'(x',y',z')$ to the point $X(x,y,z)$ is defined by parametric equations:

$$x' = x_0 + \frac{r^2(x - x_0)}{w}, \quad y' = y_0 + \frac{r^2(y - y_0)}{w},$$
$$z' = z_0 + \frac{r^2(z - z_0)}{w}, \qquad (2)$$

where, $w = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$.

The properties for points, straight lines and circles are the same as in the case of CI.

The view of a spherical surface, which not passing through the centre of the director spherical surface, is again a spherical surface. On the figure 5, there is an inverse image of the spherical surface, which is inside of the DSS.

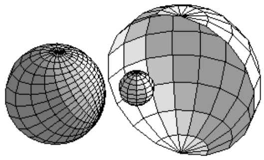

Fig. 5

If spherical surface is passing through the centre of DSS, then its image is a plane and vice versa, Fig.6.

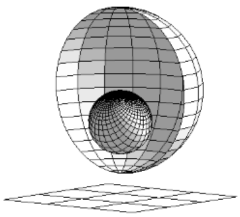

Fig. 6

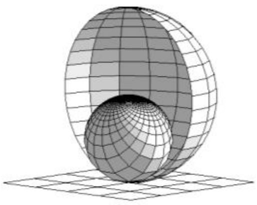

Fig. 7

On the picture 7, there is an image of the tangent plane. The plane passing through the centre of DSS is invariant, but not point invariant. The spherical surface intersecting the director spherical surface orthogonally is invariant (but not point invariant).

3 LINEAR SURFACES IN SPHERICAL INVERSION

3.1 Linear surfaces of revolution

Linear surfaces of revolution are generating by the revolution of a straight line about the axis of the revolution, where every point of the straight line is revolving on a circle lying in the plane perpendicular to the axis. Therefore, these surfaces have two systems of lines namely straight lines and circles. From the properties of the spherical inversion results, that the view of this kind of the surface have to be a surface, where its two systems of lines will be circles and at the same time one system of circles always has to pass through the centre of the director spherical surface (they are images of straight lines in SI). According to the mutual position of a straight line and an axis of revolution, we have three cases.

3.1.1 Cylindrical surface of revolution

The straight line parallel to the axis of revolution creates by revolution a cylindrical surface of revolution, which inverse image is in general a central horn cyclide, Fig.8. On the figure 9, there is a case, when the centre of DSS is the interior point of the cylindrical surface of revolution. The inverse view of cylindrical surface is now a special case of a central ring cyclide, where one circle is reduced to a point.

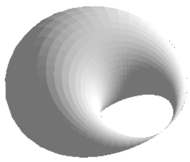

Fig. 8

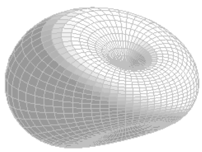

Fig. 9

If the centre of DSS is lying on a cylindrical surface, the inverse view is an axoid. It is a torus, in which every semi-meridian is passing through one point, Fig.10.

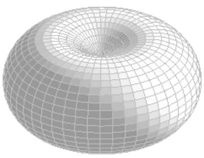

Fig. 10

Fig. 11

If one line on cylindrical surface passes trough the centre of DSS, then its image is a parabolic horn cyclide and does not matter, which is a radius of DSS, Fig.11.

3.1.2 Conical surface of revolution

If the straight line and the axis of the revolution are intersecting lines then we get conical surfaces of revolution. Its view in a spherical inversion has to be a surface, of which two frames of circles are passing through two points. The first one is the centre of DSS and the second one is the inverse view of the vertex of the conical surface. If the centre of DSS does not lie on the conical surface of revolution, then its inverse view is a central cyclide. Two cases are possible. If the centre of DSS is the exterior point of the conical surface of revolution, we get a central horn cyclide (Fig.12) and if it's the interior point, then the image is a central spindle cyclide, Fig.13. For clarity we projected only a part of this surface.

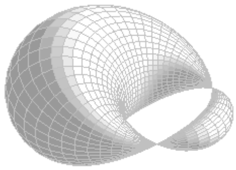

Fig. 12

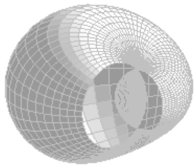

Fig. 13

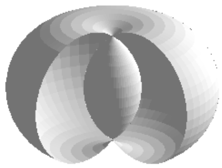

Fig. 14

Fig. 15

If the centre of the DSS is on the axis of the conical surface of revolution, the view is a melonoid, a special case of torus, whose semi-meridian circles are intersecting in two points, Fig.14. Again, we projected only one part of this surface. If the centre of DSS lies on the conical surface of revolution, then the image is a parabolic spindle cyclide (Fig.15), because the straight line passing trough centre of DSS is invariant.

3.1.3 One-sheet hyperboloid of revolution

By the rotation of a skew line about axis of the revolution, we get a one-sheet hyperboloid of revolution. Through every point of this hyperboloid two straight lines are passing (1^{st} and 2^{nd} frame) and a latitudinal circle located in the plane perpendicular to the axis of revolution. Therefore, in general, by every point of a surface, which is an inverse view of a one-sheet hyperboloid of revolution, are passing three circles. The two, which are inverse views of straight lines, have to pass through the centre of DSS.

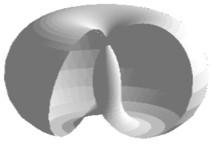
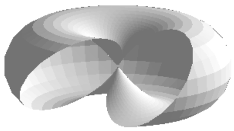

Fig. 16 Fig. 17

If the centre of DSS is on the axis of one-sheet hyperboloid of revolution, the image is a globoid, Fig. 16. Globoid is a surface, which is created by revolution of a circle about the axis, which is not located in the plane of a circle [3]. If the centre of DSS is identical with the centre of hyperboloid, the inverse view is a special globoid, which is an analogy of the axoid between tori, Fig.17.

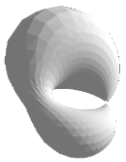
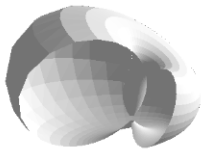
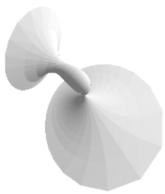

Fig. 18 Fig. 19 Fig. 20

On the picture 18, we illustrated a surface, where the centre of DSS is an exterior point of a hyperboloid. Picture 19 is a demonstration of a spherical inversion, where the centre of DSS is an interior point of a hyperboloid. If the centre is on the hyperboloid, then three straight lines are lying on the inverse surface. Two are generating lines, which are passing through the centre of DSS and the third is an inverse view of a latitudinal circle passing through this point.

3.2. Hyperbolic paraboloid

Hyperbolic paraboloid is a warped linear surface, which is formed by two frames of straight lines. Straight lines of both frames are mutually skew and lie in parallel surfaces. Through each point of a surface passes a straight line from the first and the second frame of straight lines. From the definition of a spherical inversion results, that inverse view of a hyperbolic paraboloid has to be a surface created by two systems of circles, where all circles are passing through the centre of DSS.

For better imagination, for the spherical inversion of this surface we will illustrate also their mutual position and independently also the resulting non-spherical surface. If DSS hasn`t a special position, then the inverse surface is displayed on the picture 21.

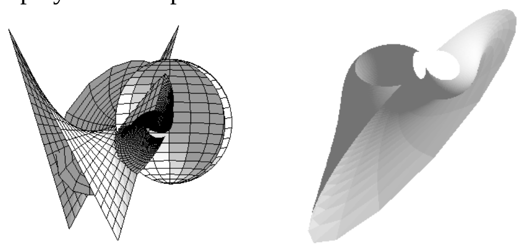

Fig. 21

If the centre of the DSS lies on the straight line passing through the saddle point and is perpendicular to the tangent plane of a hyperbolic paraboloid (in this point), we acquire interesting surfaces.

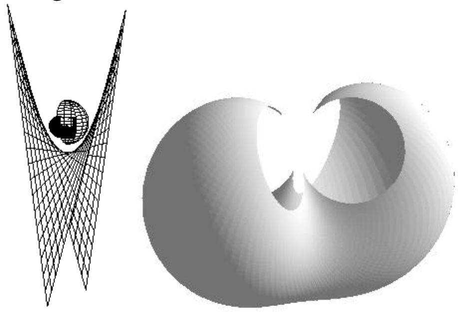

Fig. 22

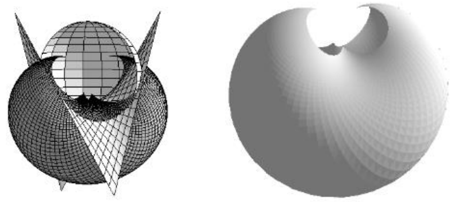

Fig. 23

There are three cases, which may come into consideration. The DSS is not intersecting the hyperbolic paraboloid (Fig.22), DSS is intersecting the hyperbolic paraboloid (Fig.23), or DSS is touching to the hyperbolic paraboloid. It can touch on the saddle point or on two points, an inverse view of this surface is a type illustrated in Fig.22.

4 CONCLUSION

Our aim was to show the existence of non-spherical surfaces, which have two systems of circles and they are neither non-spherical surfaces nor torus. We made use of a spherical inversion, in which we have displayed the linear surfaces, whose determining lines are straight lines or straight lines and circles.

Known examples of non-spherical surfaces are cyclides. We have shown only some types of cyclides, which are an inverse image of linear surfaces of revolution. All types of central and parabolic cyclides [1] could be obtained by a torus transformation in the spherical inversion [2]. The mutual position of a DSS and a torus defines the type of a cyclide.

The before mentioned surfaces were illustrated by using the software MAPLE and for the mapping of these surfaces we used parametric equations of surfaces.

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