MATHEMATICAL MODEL OF INVERTED PENDULUM WITH AIRCRAFT PROPERTIES

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The objective of this paper is to demonstrate the design of the inverted pendulum model with aircraft properties. The inverted pendulum is inherently unstable system and it is necessary to stabilize the pendulum closed loop system with PID regulator. Regulated parameter is force applied to the pendulum and regulation parameter is deviation from vertical position θ. The linearized model of an inverted pendulum is used to adjust derivation and proportional coefficients of the PID regulator, integration parameter is zero in this case. This regulator will be also tested on nonlinear model of the inverted pendulum.

K e y w o r d s. Inverted pendulum, thrust vectored aircraft, PD regulator

1 INTRODUCTION

Assuming, an aircraft is equipped with thrust vectored nozzles in low dynamic pressure conditions. In these conditions the aerodynamic forces is negligible and only moments and forces generated by thrust vectoring nozzle system is considering. Parameters of the interest are aircraft mass *m*, moment of inertia *I* and arm *l* (distance between CG and vectored nozzle). Selected parameters of simulated aircraft have following values [2]:

$m=15180\ kg;$
$I=4.2138\cdot 10^5\ kg\ m^2$
$l=5.4\ m$

These parameters are constant during simulation. The inputs is force generated by thrust for controlling purpose and its value is given by angle of deflection of the nozzle denoted φ. Angle and rate of deflection is limited and nozzle dynamic is described by following 2nd order transfer function [1]:

$$\frac{400}{s^2+40s+400}$$

Nozzle dynamic will be replaced by step function and only position limit (±20 deg) and rate limit (60 deg/sec) will be taken into account.

Time response of the inverted pendulum (red line) with same values *m, I, l* like thrust vectored aircraft (blue line), shows Figure 1.

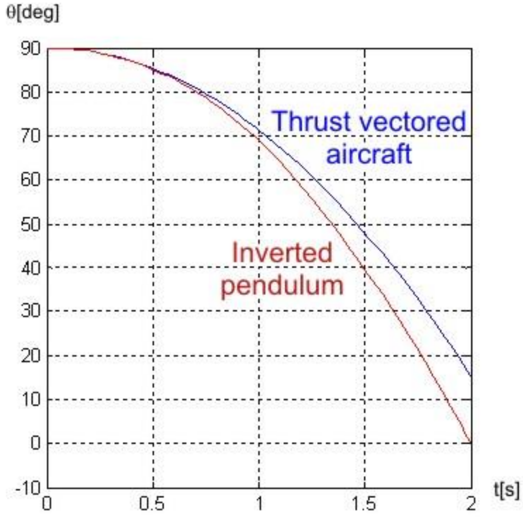

Figure 1. Time responses of the inverted pendulum and thrust vectored aircraft

2 MATHEMATICAL MODEL OF INVERTED PENDULUM

The inverted pendulum apparatus is a system with mass above its pivot point mounted on the cart. The cart can move horizontally. Depicted inverted pendulum in coordinate system is given in Figure 2.

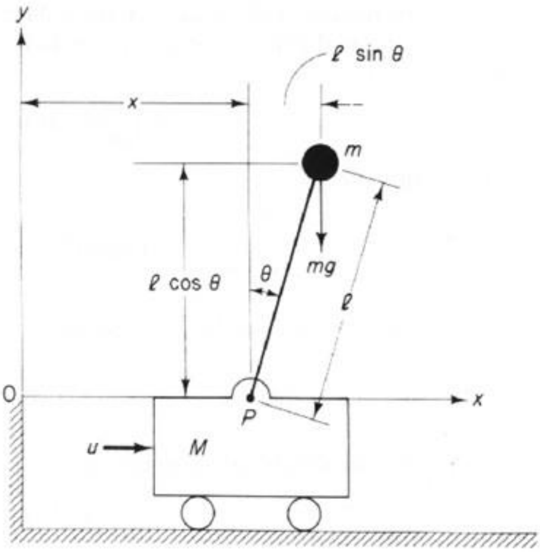

Figure 2. Inverted pendulum in coordinate system [3]

The dynamic of the pendulum is described by following equations representing nonlinear model of inverted pendulum:

$$M\frac{d^2}{dt^2}x + m\frac{d^2}{dt^2}x_G + b\frac{d}{dt}x = u \qquad (1)$$

$$I\frac{d^2}{dt^2}\theta + \left(F_x\cos\theta\right)l - \left(F_y\sin\theta\right)l = mgl\sin\theta \qquad (2)$$

$$x_G = x + l\sin\theta; \quad y_G = l\cos\theta$$

$$F_x = m\frac{d^2}{dt^2}x_G; \quad F_y = m\frac{d^2}{dt^2}y_G$$

where M – mass of the cart, m – mass of the pendulum, b – friction of the cart, l – length to pendulum centre of mass, I – inertia of the pendulum.

x and y are exact functions of theta and can be represented by their derivates in terms of theta derivates:

$$x_G = x + l\sin\theta$$

$$\frac{d}{dt}x_G = \frac{d}{dt}x + l\frac{d}{dt}\theta\left(\cos\theta\right)$$

$$\frac{d^2}{dt^2}x_G = \frac{d^2}{dt^2}x + l\frac{d^2}{dt^2}\theta\left(\cos\theta\right) - l\left(\frac{d\theta}{dt}\right)^2\sin\theta \qquad (3)$$

$$y_G = l\cos\theta$$

$$\frac{d}{dt}y_G = -l\frac{d}{dt}\theta\left(\sin\theta\right)$$

$$\frac{d^2}{dt^2}y_G = -l\frac{d^2}{dt^2}\theta\left(\sin\theta\right) - l\left(\frac{d\theta}{dt}\right)^2\cos\theta \qquad (4)$$

Substituting equations (3), (4) to equations (1), (2):

$$\left(M+m\right)\frac{d^2x}{dt^2} + ml\frac{d^2\theta}{dt^2}\cos\theta - ml\left(\frac{d\theta}{dt}\right)^2 = u \qquad (5)$$

$$\left(I+ml^2\right)\frac{d^2\theta}{dt^2} = -ml\frac{d^2x}{dt^2}\cos\theta + mgl\sin\theta \qquad (6)$$

Equations (5), (6) representing mathematical model of an inverted pendulum depicted on Figure 2. For PD regulator design it is necessary to linearize these equations in position when θ=0; where the mass of the pendulum is exactly above a cart pivot point and θ represents small deviation from this position. Let's assume following simplifications [4]:

$$\cos\theta = 1;\ \sin\theta = \theta;\ \left(\frac{d\theta}{dt}\right)^2 = 0$$

Linearized equations became:

$$\left(M+m\right)\ddot{x} + b\dot{x} + ml\ddot{\theta} = u \qquad (7)$$

$$\left(I+ml^2\right)\ddot{\theta} = -ml\ddot{x} + mgl\theta \qquad (8)$$

Applying Laplace transformation, equations (7) and (8) have following form:

$$\left(M+m\right)X\left(s\right)s^2 + bX\left(s\right)s + ml\theta\left(s\right)s^2 = U\left(s\right) \qquad (9)$$

$$\left(I+ml^2\right)\theta\left(s\right)s^2 = -mlX\left(s\right)s^2 + mgl\theta\left(s\right) \qquad (10)$$

X(s) can be calculated from equation (10):

$$X\left(s\right) = \left(\frac{g}{s^2} - \frac{I+ml^2}{ml}\right)\theta\left(s\right) \qquad (11)$$

Then substituting (11) to equation (9) and re-arranging, the final transfer function is:

$$\frac{\theta\left(s\right)}{U\left(s\right)} = \frac{\dfrac{ml}{\left(ml\right)^2 - \left(M+m\right)\left(I+ml^2\right)}}{s^2 + \dfrac{mgl\left(M+m\right)}{\left(ml\right)^2 - \left(M+m\right)\left(I+ml^2\right)}} \qquad (12)$$

For similarity with aircraft assume, that friction coefficient b=0 and mass of the cart is negligible, because its value is in relation of pendulum mass much smaller. Using previous simplifications the transfer function is:

$$\frac{\theta(s)}{U(s)}=\frac{\dfrac{ml}{(ml)^2-m\left(I+ml^2\right)}}{s^2+\dfrac{m^2gl}{(ml)^2-m\left(I+ml^2\right)}} \tag{13}$$

General form of transfer function (13) is:

$$\frac{\theta(s)}{U(s)}=\frac{K}{s^2+\omega_0^2} \tag{14}$$

In this case, the parameter K and natural frequency of the system ω_0 are given:

$$\omega_0^2=-g\frac{ml}{I};\ K=-\frac{l}{I} \tag{15}$$

You can see the results depend only on ratio between constants. This is very important fact for further design so it is not necessary to know exact value of the parameters. It is possible to replace this system by other system with the same ratio between the parameters for natural frequency to simulate same dynamic. This idea is useful for practical model of inverted pendulum with aircraft properties implementation. The outputs parameter of this system will be same, only value of mass, inertia, arm and input force will be much smaller. This system is expected to utilize for practical control system testing.

3 PID REGULATOR DESIGN

It is easy way for further design apply transfer function in form given by equation (14). PD regulator for pendulum stabilization is described by following expression:

$$Ds+P \tag{16}$$

where P is proportional and D is derivation coefficient.

Figure 3 depicts block diagram of the closed loop feedback control system in general.

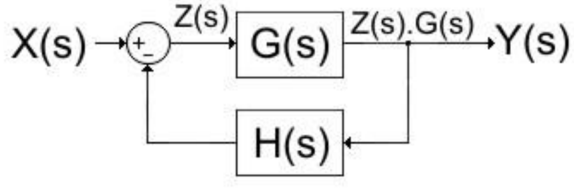

Figure 3. Closed loop system

Closed loop transfer function has form [5]:

$$\frac{Y(s)}{X(s)}=\frac{G(s)}{1+H(s)\cdot G(s)} \tag{17}$$

Z(s) is input to the plant and in this example its value is limited. This value cannot exceed the limitation. Use following transfer function to observe the input:

$$\frac{Z(s)}{X(s)}=\frac{1}{1+H(s)\cdot G(s)} \tag{18}$$

In this example θ(s) represents Y(s), X(s) can by replaced by U(s) and G(s) is inverted pendulum transfer function equation (14). H(s) describes PD given by (16). Final form of closed loop system is calculated by substituting equations (14) and (16) to (17):

$$\frac{\theta(s)}{U(s)}=\frac{K}{s^2+KDs+\left(PK+\omega_0^2\right)} \tag{19}$$

Compare denominator of equation (19) with binomial standard form for 2nd order system. Parameters of PD regulator are:

$$\begin{aligned}P&=-\left(\frac{I}{l}\omega_z^2+mg\right);\\ D&=-\frac{2I}{l}\omega_z;\end{aligned} \tag{20}$$

where ω_z is desired natural frequency of system. For PD regulator design, it is important to state gain and time of regulation. The value of angle θ can be calculated in $t\to\infty$:

$$\theta(s)=\frac{K}{PK+\omega_0^2}U(s)=-\frac{l}{I\omega_z^2}U(s) \tag{21}$$

For approximate calculation of time of regulation t_r of 2nd order system utilize formula:

$$t_r=\frac{7}{\omega_z}$$

4 SIMULATION RESULT

Selected parameters *m*, *I*, *l* of simulated aircraft were mentioned above. Substitute these values into equation (14). Transfer function has following form:

$$\frac{\theta(s)}{U(s)} = \frac{-1.2815 \cdot 10^{-5}}{s^2 - 1.90836} \qquad (22)$$

PD regulator for the pendulum is connected into feedback of closed loop system depicted on Figure 3. Apply equation (20) to calculate parameters P and D of the regulator. State that desired value of natural frequency is 2 rad/sec. Calculated parameters P, D are:

$$P = -461049$$
$$D = -312133$$

Assume that engines of aircraft produce thrust 148916 N and deflection of the vectored nozzle is limited to ±20 deg, so maximum value of thrust to control the aircraft is 50932 N [1] [2].

Time response on the input step function is on Figure 4. Apply equation (21) for angle θ calculation in t→∞:

$$\theta(s) = -\frac{l}{I\omega_z^2} U(s) = \frac{5.4 \cdot 50932}{4.2138 \cdot 10^5 \cdot 2^2} \cong 9°21'$$

$$t_r = \frac{7}{2} = 3.5 \text{ sec}$$

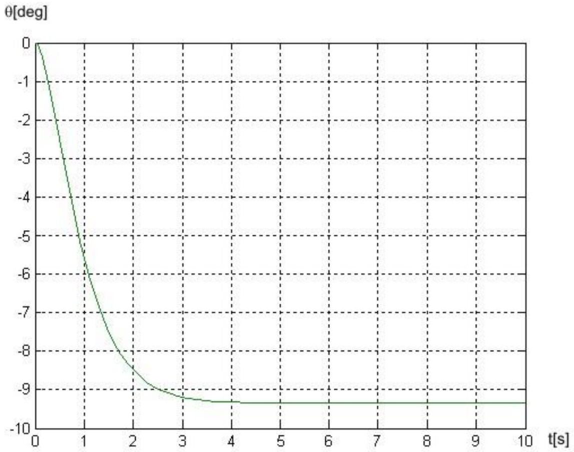

Figure 4. θ angle time response

Figure 5 depicts force in the input to the pendulum. You can see that limited value has not been exceeded.

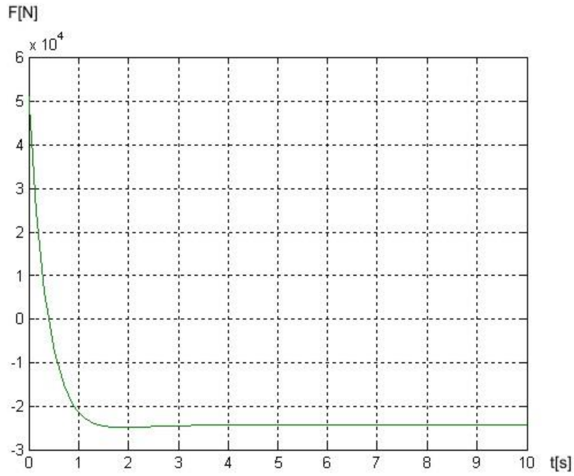

Figure 5. Force applied to the pendulum

Comparison between linear model described by transfer function (22) and nonlinear model represented by equations (5), (6) is on the Figure 6.

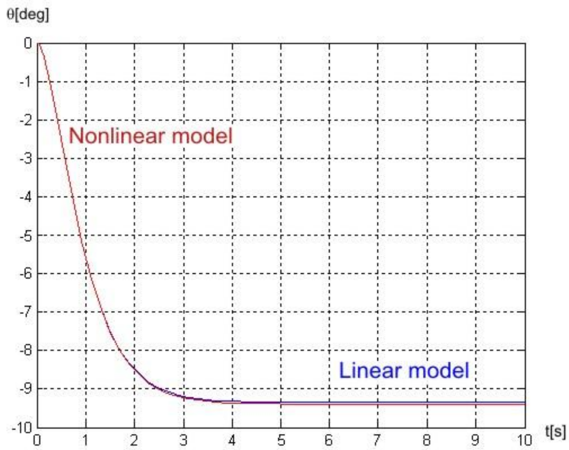

Figure 6. Comparision between linear and nonlinear model

5 CONCLUSION

It is possible to apply PD regulator for inverted pendulum stabilization. The advantage of this system is relatively simply structure. Design of this system and PD coefficients calculation is also not complicated and linear transfer function can be utilized for design. The differences between linear and nonlinear model are small, final results correspond to the calculation and the comparison between these models is shown on Figure 6. Disadvantage is relatively small range of regulated angles and the fact, that out of this range the motion of the pendulum becomes uncontrolled.

BIBLIOGRAPHY

[1] ADAMS, Richard J. – BUFFINGTON, James M. – SPARKS, Andrew G. – BANDA, Siva S.: Robust Multivariable Flight Control. London: Springer-Verlag, 1994. ISBN 3-540-19906-3

[2] Praktická aerodynamika lietadla MiG-29

[3] http://www.profjrwhite.com/system_dynamics/sdyn/s7/s7invp1/s7invp1.html#equations

[4] http://www.me.cmu.edu/ctms/controls/ctms/matlab42/examples/pend/invpen.htm

[5] http://en.wikipedia.org/wiki/Closed-loop_transfer_function

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