CYCLIDES AS A SPECIAL TYPE OF CYCLICAL ELLIPTICAL PEDAL SURFACES

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Intensive use of geometrical modelling started with the development of computer graphics and CAD/CAM technologies, comprising a wide area of mathematical disciplines. Geometrical modelling is a synthesis of the geometry and computer graphics, which enable us to develop complex mathematical models that would be rather difficult to display without using a computer. We recognize two basic research methods of geometrical models, the synthetic and the analytic one. The former method makes use of geometrical constructions whereas the latter characterizes geometrical objects by numerical data. Both methods have been applied in this contribution.

The aim of this research is to show central cyclides (Dupin´s cyclides) as a special kind of the cyclical elliptical pedal surfaces. Firstly, using the method of synthetic geometry, a new class of such surfaces in the 3-dimensional Euclidean space (model of the projective space) is defined. Geometrical construction of these surfaces is dependent on the given ellipse and the position of the pole *P*. It is the point at which the pencil of perpendicular planes passes to the plane of the ellipse, wherein the generating circles of the surface are lying. The parametric equations of the cyclical elliptical surface are derived applying the method of analytic geometry. We classify the surfaces according to the number of generating circles with a zero radius. The evolute of the ellipse divides the plane of the ellipse into two areas, Ω_1 and Ω_2. The shape of the surface depends on the position of the pole *P* in the areas of Ω_1 and Ω_2. A surface can have 4, 3 or 2 circles with a zero radius. We then describe the relationship between these surfaces and the Dupin´s cyclides, obtained in the case when the pole *P* is the point on the major axis of the ellipse. Finally, the transformation of these surfaces is shown by changing of the orthonormal base in the parameterization of the generating circles of the surface. The resulting surfaces obtained by the parametric approach are visualized in the MAPLE program environment.

K e y w o r d s. cyclical elliptical pedal surfaces, creation, classification, Dupin´s cyclides

1 INTRODUCTION

Several mathematicians attended to the investigation of cyclides by different ways [1], [2], [3], [4]. A survey of various definitions of cyclide, as well as a description of the properties of different shapes of cyclides can be found in [5], [6]. Our aim is to show the cyclides as a special type of cyclical elliptical pedal surfaces. We use the analytic geometry of curves and surfaces which is well described in [7], [8] and [9].

2 CREATION OF CYCLICAL ELLIPTICAL PEDAL SURFACE AND ITS PARAMETERIZATION

2.1 Geometrical way of surface creation

Let us have the ellipse κ and the point *P* in the plane π. Let *K* be the point of the ellipse different from a main vertex. F_1, F_2 are the foci of ellipse κ. We construct accompanying lines of the point *K* and by the point *P* perpendicular line *r* to the tangent *t* of ellipse at the point *K*. We denote $L = r \cap t$, $K_1 = r \cap F_1K$ and $K_2 = r \cap F_2K$. L is the midpoint of $|K_1K_2|$ and currently the point on the pedal curve of the ellipse κ for the pole *P* (see Fig.1).

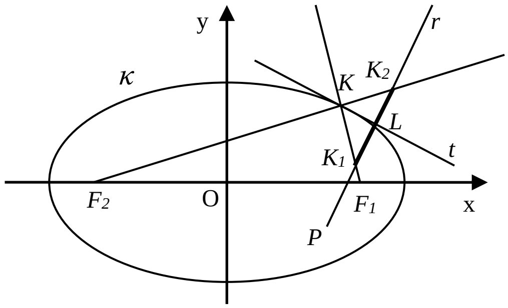

Fig.1 Situation in the plane π

Let *k* be the circle with the centre *L* passing through points K_1, K_2 lying in the plane perpendicular to the plane π. The circle *k* reduces to a point (circle with zero radius) if the line *r* is the normal line of ellipse κ. If the point *K* moves along the ellipse (except main vertices) then by the above described method we get the system of circles that are generating circles of cyclical elliptical pedal surface.

The situation in the main vertices is as follows: The accompanying lines of the main vertices coincide with the major axis of the ellipse. There are two cases:

(i) If P is not the point of the major axis of ellipse then a line r is parallel to the major axis. Corresponding points K_1, K_2 are points at infinity and the cyclical elliptical pedal surface is not bounded. This case is discussed in the first chapter of the paper.

(ii) If P is the point of the major axis of ellipse then the line r and the accompanying lines of the foci coincide with the major axis of the ellipse. The intersection points of the line r with accompanying lines are points at infinity and the cyclical elliptical pedal surfaces are bounded. It is shown in the second chapter.

2.2 Parameterization of surface

Let the Cartesian coordinate system (O; x, y) in the plane π be the following: the origin O is the centre of ellipse κ, coordinate axis x is a major axis of ellipse and y is a minor axis of ellipse, Fig.1. As usual we label as a semi-major axis, b the semi-minor axis and e the eccentricity of an ellipse.

An ellipse will be parameterized by a parameter u, which is an arc of the spherical view of the normal of ellipse κ. An ellipse is parameterized by vector function

$$\boldsymbol{\kappa}(u) = \left(\frac{a^2 \cos u}{h(u)}, \frac{b^2 \sin u}{h(u)} \right), \quad u \in [0, 2\pi[, \qquad (1)$$

where function $h(u) = \sqrt{a^2 \cos^2 u + b^2 \sin^2 u}$, $u \in [0, 2\pi[$ is the support function of the ellipse. Unit direction vectors of normal lines of the ellipse are given by vector function

$$\mathbf{n}(u) = (\cos u, \sin u), \quad u \in [0, 2\pi[. \qquad (2)$$

Let P be the point $P = [x_0, y_0]$, $y_0 \neq 0$. Then for a given value of the parameter u the line r has the parametric expression

$$x = x_0 + t \cos u, \quad y = y_0 + t \sin u, \quad t \in R. \qquad (3)$$

Foci are $F_1 = [e, 0]$ and $F_2 = [-e, 0]$. Accompanying lines $F_1 K(u)$ and $F_2 K(u)$ are given by

$$b^2 \sin u\, x + \left(-a^2 \cos u + e h(u)\right) y = b^2 e \sin u, \qquad (4)$$

and

$$b^2 \sin u\, x - \left(a^2 \cos u + e h(u)\right) y = -b^2 e \sin u. \qquad (5)$$

Substituting Eq. (3) into Eqs. (4) or (5), we express the value of parameter t on the line r for the point $K_1(u)$, or $K_2(u)$. Values $t_1(u)$ and $t_2(u)$ are denoted as the appropriate ones of a parameter and we get

$$t_1(u) = \frac{b^2 (e - x_0) \sin u - y_0 \left(e\, h(u) - a^2 \cos u\right)}{e \left(h(u) - e \cos u\right) \sin u}, \qquad (6)$$

$$t_2(u) = \frac{b^2 (e + x_0) \sin u - y_0 \left(e\, h(u) + a^2 \cos u\right)}{e \left(h(u) + e \cos u\right) \sin u},$$

$$u \in \left]0, \pi\right[\cup \left]\pi, 2\pi\right[. \qquad (7)$$

If $u = 0$ and $u = \pi$ then we get the major vertices, points $K(0)$ and $K(\pi)$. Provided that $y_0 \neq 0$, it follows from Eqs. (6) and (7) that if $u = 0$ and $u = \pi$, the functions t_1, t_2 are not defined. The limits of both functions for $u \to 0, u \to \pi$ are at infinity.

The parameterization of a surface means to parameterize its generating circles, thus parameterize the loci of their centers and to determine the radii functions. Centers $L(u)$ of generating circles are points of the pedal curve of the ellipse $\mathcal{L}$ for the pole P.

Pedal curve $\mathcal{L}$ is parameterized by point function

$$\mathbf{L}(u) = \left[x_0 + \frac{1}{2}\left(t_2(u) + t_1(u)\right) \cos u,\ y_0 + \frac{1}{2}\left(t_2(u) + t_1(u)\right) \sin u \right]$$

Substituting (6) and (7) we get

$$\mathbf{L}(u) = \begin{bmatrix} x_0 + \left(h(u) - x_0 \cos u - y_0 \sin u\right) \cos u, \\ y_0 + \left(h(u) - x_0 \cos u - y_0 \sin u\right) \sin u \end{bmatrix},$$

$$u \in [0, 2\pi[. \qquad (8)$$

Radii of generating circles are the values of function

$$R(u) = \frac{1}{2}\left|K_1(u)K_2(u)\right| = \frac{1}{2}\left|t_2(u) - t_1(u)\right| =$$

$$= \left|\frac{h(u)(x_0 \sin u - y_0 \cos u) - e^2 \sin u \cos u}{e \sin u}\right|,$$

$$u \in \left]0, \pi\right[\cup \left]\pi, 2\pi\right[. \qquad (9)$$

Function in Eq. (9) is not defined for $u = 0$ and $u = \pi$. The limits of function R for $u \to 0$ and $u \to \pi$ are at infinity, therefore the surface is not bounded.

The cyclical elliptical pedal surface is parameterized by the point function

$$\mathbf{X}(u,v) = \mathbf{L}(u) + R(u)(\cos v\,\mathbf{n}(u) + \sin v\,\mathbf{e}_3),$$

$$u \in \left]0, \pi\right[\cup \left]\pi, 2\pi\right[, v \in \left[0, 2\pi\right[, \qquad (10)$$

in which *R(u)* is a real function in Eq. (9), $\mathbf{e}_3 = (0,0,1)$ and functions **L**(u), **n**(u) are extended by third zero coordinate.

2.3 The loci of points K1 and K2

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the loci of the points $K_1(u)$ and $K_2(u)$, where $u \in \left]0, \pi\right[\cup \left]\pi, 2\pi\right[$. Coordinates of the points $K_1(u)$ and $K_2(u)$ are values of functions

$$x(u) = x_0 + t_1(u)\cos u, \quad y(u) = y_0 + t_1(u)\sin u \qquad (11)$$

and

$$x(u) = x_0 + t_2(u)\cos u, \quad y(u) = y_0 + t_2(u)\sin u, \qquad (12)$$

where t_1, t_2 are functions, see Eqs. (6) and (7).

Substituting Eq. (6) into Eq. (11) or Eq. (7) into Eq. (12) and after editing

$$x(u) = \frac{(x_0 \sin u - y_0 \cos u)(eh(u) - a^2 \cos u) + b^2 e \sin u \cos u}{e \sin u (h(u) - e \cos u)}$$

$$y(u) = \frac{b^2(-x_0 \sin u + y_0 \cos u + e \sin u)}{e(h(u) - e \cos u)} \qquad (13)$$

and

$$x(u) = \frac{(x_0 \sin u - y_0 \cos u)(eh(u) + a^2 \cos u) + b^2 e \sin u \cos u}{e \sin u (h(u) + e \cos u)}$$

$$y(u) = \frac{b^2(x_0 \sin u - y_0 \cos u + e \sin u)}{e(h(u) + e \cos u)},$$

$$u \in \left]0, \pi\right[\cup \left]\pi, 2\pi\right[. \qquad (14)$$

It is evident that for $u \to 0\ (u \to 2\pi)$ or $u \to \pi$, then the limit of the first coordinate function *x(u)* in Eq. (13) is at infinity. It means that the points are $K_1(0)$ or $K_2(\pi)$ are at infinity. The second coordinate function of the points $K_1(0)$ and $K_2(\pi)$ is a number

$$y = \frac{y_0 b^2}{e(a-e)} \text{ and } y = \frac{-y_0 b^2}{e(a+e)}. \qquad (15)$$

Equally for $u \to 0\ (u \to 2\pi)$ and $u \to \pi$ the limit of the first coordinate function *x(u)* in Eq. (14) is at infinity and therefore points $K_1(0)$ and $K_2(\pi)$ are at infinity. The second coordinate function of the points $K_1(0)$ and $K_2(\pi)$ is a number

$$y = \frac{-y_0 b^2}{e(a+e)} \text{ and } y = \frac{y_0 b^2}{e(a-e)}. \qquad (16)$$

The locus of points $\mathcal{K}_1$ ($\mathcal{K}_2$) fall into two curves. The first one is for $u \in]0, \pi[$ and the second one for $u \in]\pi, 2\pi[$. Lines with equations (15) or (16) are asymptotes of all curves, which we get for different positions of the point $P = [x_0, y_0]$, $y_0 \neq 0$. In Fig.2 (Fig.3) we illustrate the locus $\mathcal{K}_1$ ($\mathcal{K}_2$) for the choice $a = 5$, $b = 3$, $P = [0.5, 2]$.

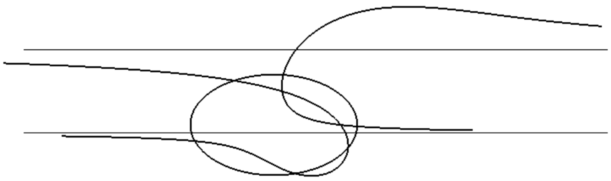

Fig. 2 Disintegrated curve $\mathcal{K}_1$ with asymptotes

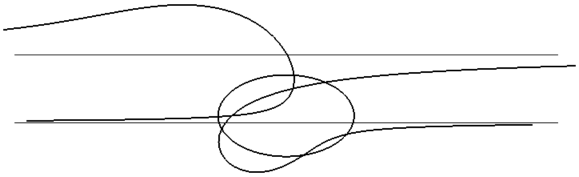

Fig. 3 Disintegrated curve $\mathcal{K}_2$ with asymptotes

If a line *r* is the normal of the ellipse passing through the point *P*, then $K_1 = K_2$ is the point of ellipse κ. Such points are exactly as many as intersection points of ellipse normal passing trough the point *P* with the ellipse. The number of intersection points depends on the position of the point *P* with regard to the evolute $\mathcal{M}$ of the ellipse κ. The evolute is parameterized by the point function

$$\mathbf{M}(u) = \left[\frac{a^2 \cos u}{h(u)} - \frac{a^2 b^2 \cos u}{h^3(u)}, \frac{b^2 \sin u}{h(u)} - \frac{a^2 b^2 \sin u}{h^3(u)}\right], \quad u \in [0, 2\pi[\quad (17)$$

and the equation of the evolute is

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (e^2)^{\frac{2}{3}}. \quad (18)$$

The evolute $\mathcal{M}$ divides the plane π into two areas. The area Ω_1 and Ω_2 (see Fig.4) is the set of the points whose coordinates satisfy the inequality

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} < (e^2)^{\frac{2}{3}} \text{ and } (ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} > (e^2)^{\frac{2}{3}}.$$

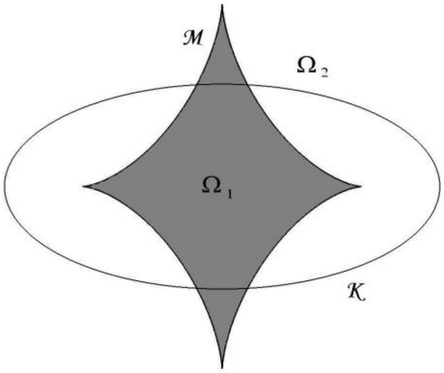

Fig. 4 Divided plane

In case that *P* is the point of an area Ω_1, then there exist 4 intersection points of the normals of ellipse that are passing through the point *P* and the ellipse. Therefore, we obtain 4 points for which $K_1 = K_2$.

If *P* is the point of evolute $\mathcal{M}$, its coordinates satisfy the equation (18), and there exist 3 intersection points of normals to the ellipse passing through the point *P* and the ellipse κ, if *P* is not a singular point of the evolute on a minor axis of an ellipse. If *P* is a singular point there exist only two intersection points. In case that the point *P* is the point of the area Ω_2, then there exist 2 intersection points.

2.4 Classification of cyclical elliptical pedal surfaces

We will describe surfaces according to a number of generating circles with a zero radius, thus points. Radii of generating circles are values of a function Eq. (9). It is evident that a function has a zero value if and only if for some value $u \in]0, \pi[\cup]\pi, 2\pi[$ is $K_1(u) = K_2(u)$. So the cyclical elliptical pedal surface has

(i) 4 generating circles with a zero radius if *P* is the point of the area Ω_1. In Fig.5 we illustrate a section of the surface for $a = 5,\ b = 3,\ P = [0.5, 1]$.

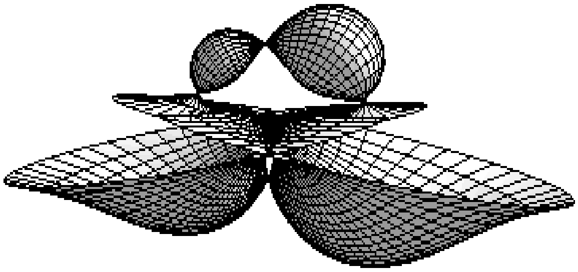

Fig. 5 Cyclical elliptical pedal surface with 4 zero generating circles

(ii) 3 generating circles with a zero radius if P is the point of the evolute $\mathcal{M}$ but not a singular point on the minor axis of an ellipse (see Fig.6).

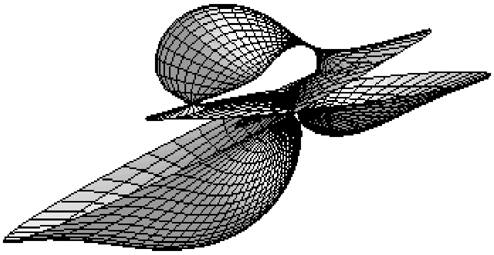

Fig. 6 Cyclical elliptical pedal surface with 3 zero generating circles

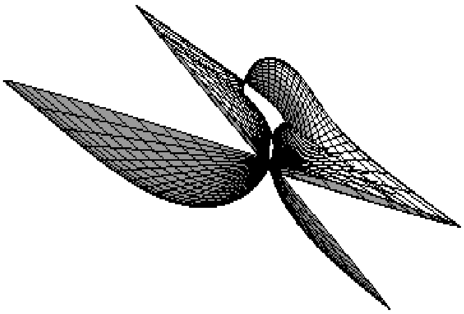

Fig. 7 Cyclical elliptical pedal surface with 2 zero generating circles

(iii) 2 generating circles with a zero radius if P is the point of the area Ω_1 or the singular point of the evolute on the minor axis of an ellipse. In Fig.7 we illustrate a section of the surface for $a=5,\ b=3,\ P=[4.2, 5]$.

3 CONNECTION OF CYCLICAL ELLIPTICAL PEDAL SURFACES WITH DUPIN´S CYCLIDES

So far we assumed that the point P is not the point on a major axis of an ellipse. In this chapter we will assume, that P is the point on the major axis. Then $P=[x_0, 0]$. Substituting $y_0=0$ into Eqs. (13) or (14), we evaluate the coordinates of the points K_1 and K_2. Their coordinates are

$$x(u)=\frac{x_0\left(e\,h(u)-a^2\cos u\right)+b^2 e\cos u}{e\left(h(u)-e\cos u\right)},$$

$$y(u)=\frac{b^2\left(e-x_0\right)\sin u}{e\left(h(u)-e\cos u\right)} \tag{19}$$

and

$$x(u)=\frac{x_0\left(eh(u)+a^2\cos u\right)+b^2 e\cos u}{e\left(h(u)+e\cos u\right)},$$

$$y(u)=\frac{b^2\left(e+x_0\right)\sin u}{e\left(h(u)+e\cos u\right)}. \tag{20}$$

In this case, the function of the first coordinate in Eqs. (19) and (20) is continuous on the interval $[0, 2\pi[$, so the limit of the function for $u\to 0$ and $u\to\pi$ equals to functional value, it is a real number. Therefore, points $K_1(0), K_2(0)$ and $K_1(\pi), K_2(\pi)$ are real and the surface is bounded. For each parameter $u\in[0, 2\pi[$ is the distance $\left|F_1K_1(u)\right|=\frac{a\left|e-x_0\right|}{e}$ and $\left|F_2\,K_2(u)\right|=\frac{a\left|e+x_0\right|}{e}$.

So the set $\mathcal{K}_1$ ($\mathcal{K}_2$) is a circle, with the center at the focus F_1 (F_2) of an ellipse and its radius is $\frac{a\left|e-x_0\right|}{e}$ and $\frac{a\left|e+x_0\right|}{e}$. It means, that this cyclical elliptical pedal surface is a central Dupin´s cyclide [9] parameterized by the point function in Eq. (10), in which the point function is

$$\mathbf{L}(u)=\left[x_0+\left(h(u)-x_0\cos(u)\right)\cos u,\left(h(u)-x_0\cos u\right)\sin u, 0\right] \tag{21}$$

and the real function is

$$R(u) = \frac{1}{e}\left| x_0\, h(u) - e^2 \cos u \right|$$
$$u \in [0,\, 2\pi[,\ \ v \in [0,\, 2\pi[\,. \qquad (22)$$

Central cyclides are surfaces, which have two systems of circles, which are also lines of the curvature and two planes of symmetry. The first one is the plane π and the second one passes through the major axis of an ellipse perpendicular to the plane π. We show different types of a cyclides depending on the position of the point *P*. In the following surfaces are $a = 5,\ b = 4$.

For $P = [0,0]$ we get a symmetric horn cyclide (Fig.8).

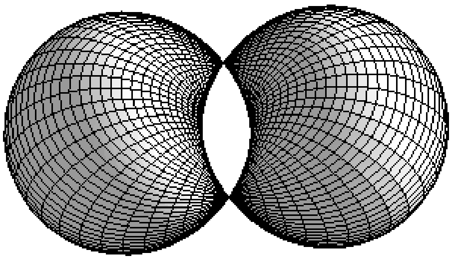

Fig. 8 Symmetric horn cyclide

When $P \subset \Omega_1$, we get a horn cyclide. For $P = [1,\, 0]$ the surface is illustrated in Fig.9.

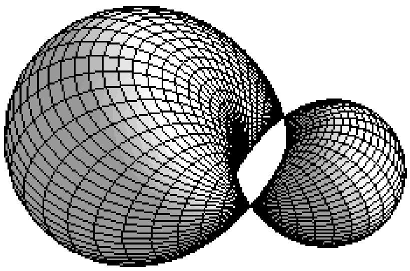

Fig. 9 General horn cyclide

If *P* is the singular point of the evolute, then the surface is a ring cyclide. Circles $\mathcal{K}_1$ and $\mathcal{K}_2$ have in plane π an inner touch (see Fig.10).

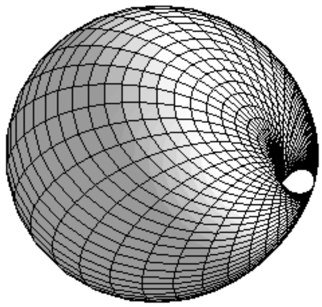

Fig. 10 Special ring cyclide

Let *P* be the point between the singular point of evolute and the focus of the ellipse, then the surface is general ring cyclide (see Fig.11).

If *P* is the focus of the ellipse then one circle is reduced to this focus. This surface is shown in Fig.12.

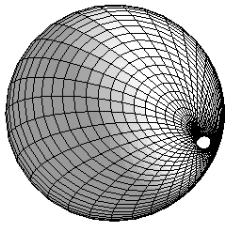

Fig. 11 Ring cyclide for *P* = [2.1,0]

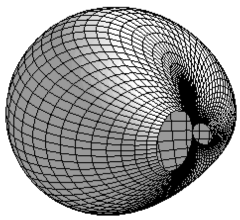

Fig. 12 Ring cyclide for *P* = [e,0]

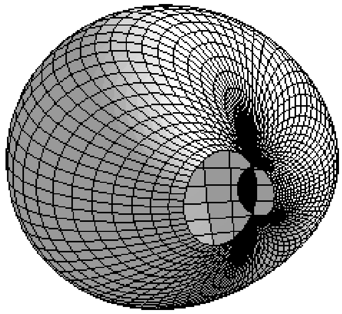

Fig. 13 Spindle cyclide

Spindle cyclide is for a pole $P \subset \Omega_2$. It is illustrated in Fig.13 for the pole $P = [4.5, 0]$.

4 TRANSFORMATION OF SURFACES

The surface is transformed by changing the orthonormal base in the parameterization of generating circles of a surface. The orthonormal base is formed by vectors $\mathbf{n}(u) = (\cos u, \sin u, 0)$, $u \in [0, 2\pi[$, $\mathbf{e}_3 = (0,0,1)$. We substitute the vector $\mathbf{e}_3$ by a vector function $\mathbf{e}(u) = \cos\alpha\, \mathbf{t}(u) + \sin\alpha\, \mathbf{e}_3$, in which α is constant, $\alpha \in \left]0, 2\pi\right[$ and $\mathbf{t}(u) = (-\sin u, \cos u, 0)$. This change corresponds to the revolution of the generating circles around diameter $K_1(u)K_2(u)$ by an angle α.

The change of the size of vectors **n**(u) and $\mathbf{e}_3$ presents an affine transformation of the surface. Generating circles are transformed into ellipses. If we transform only the size of vector $\mathbf{e}_3$, we get transformed scaled cyclides which are images of Dupin cyclides under an affine scalling application [10].

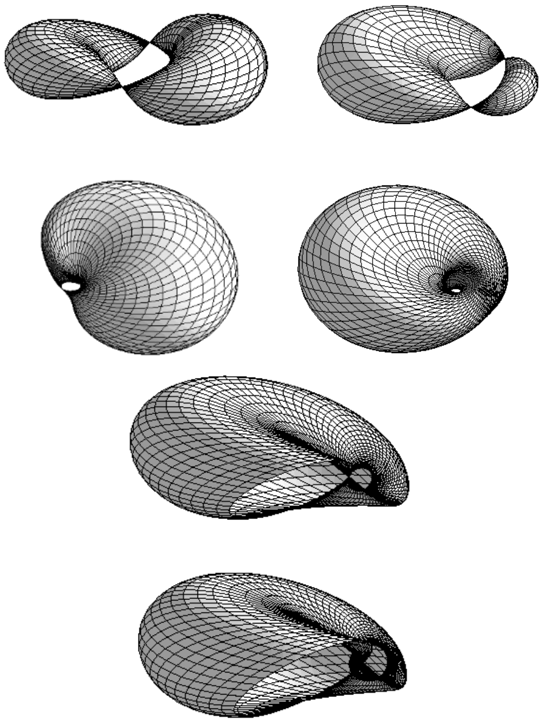

Fig. 14 Transformed cyclides

In Fig.14 there are shown transformed cyclides from Fig.8 - 13 for the angle $\alpha = \pi / 4$.

5 CONCLUSION

We have shown that Dupin´s cyclides can be viewed as a special type of cyclical elliptical pedal surfaces. Their shape depends on the position of the pole P which is the point of the major axis of the ellipse. The analytical representation of such surfaces was derived.

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