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HIT NUMBER OF DIRECTED GRAPHS

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Summary. A total coloring f of a directed graph G is called edge-irregular if for any two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ of G the associated ordered triples $(f(u_1), f(e_1), f(v_1))$ and $(f(u_2), f(e_2), f(v_2))$ are different. The problem is to determine the minimum number of colors used in such a coloring of G. This parameter of G is denoted by $hit(G)$. In this paper we survey results for this parameter.

Keywords: directed graph, graph coloring, graph labeling

1. INTRODUCTION

In a vertex coloring (labeling) of a graph G, each vertex of G is assigned a color (label). If distinct vertices are assigned distinct colors, then the coloring is called *vertex-distinguishing* or *vertex-irregular*. That is, each vertex of G is uniquely determined by its color. Similarly, an edge coloring of G is *edge-irregular* if distinct edges are assigned distinct colors.

There are occasions when a vertex coloring of a graph induces an edge-irregular labeling (edge-irregular vertex coloring for short). A proper vertex coloring is called *harmonious coloring* if every pair of colors appears on at most one pair of adjacent vertices. The harmonious coloring was introduced by Frank et al. [4] and independently by Hopcroft and Krishnamoorthy [5] in a slightly modified version, allowing that adjacent vertices may be colored with the same color (*harmonic coloring* [7]). Every harmonious of harmonic coloring f of G induces an edge labeling of G where the edges uv is assigned the label $\{f(u), f(v)\}$. Since no two edges of G are labeled the same, this vertex coloring is edge-irregular.

Jendroľ [6] generalized the harmonic coloring to *multiset coloring*. A total coloring f of a directed graph G that for any two edges the associated ordered triples are different. The problem is to determinate the minimum number of colors used in such a coloring of G. In this paper we survey results for this parameter.

2. MOTIVATIONS

Motivations for the research mentioned throughout this paper has been an intensive work on the edge irregular colouring of graphs.

3. RESULTS

3.1. General results

The following results gives upper bound for number of edges of digraph which has $hit(G)$ at most k and the fact that this property is hereditary with respect to subgraphs.

Lemma 3.1.1 [1]:
If $G=(V,E)$ is a digraph and $hit(G)\le k$, then for number of edges is $|E(G)|\le k^3$.

Lemma 3.1.2 [1]:
If $G'=(V',E')$ is a subgraph of a digraph $G=(V,E)$, then $hit(G')\le hit(G)$.

Theorem 3.1.1 [1]:
Let $G=(V,E)$, $|V|=p$, n is fixed number, $n\in\{1,\dots,\lfloor\sqrt{p}\rfloor\}$.
Then $hit(G)\le \max\left\{\left\lceil \frac{p}{n}\right\rceil, n^2\right\}$.

Proposition 3.1.1 [1]:
Let $G=(V,E)$, $|V|=p$, n is fixed number, $n\in\{1,\dots,\lfloor\sqrt{p}\rfloor\}$.
Then $hit(G)\le \min_{n\in\{1,\dots,\lfloor\sqrt{p}\rfloor\}} \max\left\{\left\lceil \frac{p}{n}\right\rceil, n^2\right\}$.

3.2 Directed bipartite graphs

Let $K_{m,n}$ denoted the directed complete bipartite graph on $m+n$ vertices, let $A=\{u_1,\dots,u_m\}$ and $B=\{v_1,\dots,v_n\}$ be the part of $K_{m,n}$ and let u_iv_j be the edges, $i\in\{1,\dots,m\}$, $j\in\{1,\dots,n\}$. We will use this notation of vertices of the two parts of $K_{m,n}$ in the whole section. Let $K^*_{m,n}$ be a graph obtained from $K_{m,n}$ by changing the directions of all edges.

Lemma 3.2.1 [2]:
$hit(K_{m,n})=hit(K^*_{m,n})$ for any $m,n\ge 1$.

Lemma 3.2.2 [2]:
$hit(K_{1,n})=\left\lfloor n^{1/2}\right\rfloor$ for any $n\ge 1$.

Lemma 3.2.3 [2]:
$hit(K_{m,n})=\left\lfloor n^{1/2}\right\rfloor$ for any $m\le\left\lfloor n^{1/2}\right\rfloor$, $n\ge 1$.

Corollary 3.2.1 [2]:
$hit(K_{m,m^2})=m$ for any $m\ge 1$.

Since Lemma 3.2.3 gives the exactly value of $hit(K_{m,n})$ for $n\ge m^2$ in the following we will assume that $n<m^2$.

Corollary 3.2.2 [2]:

$hit(K_{m,n}) \le m$ for any $n < m^2$.

Lemma 3.2.4 [2]:
If $hit(K_{m,n}) = k$, then $\left\lceil \frac{m}{k} \right\rceil \cdot \left\lceil \frac{n}{k} \right\rceil \le k$.

Lemma 3.2.5 [2]:
If $\left\lceil \frac{m}{k} \right\rceil \cdot \left\lceil \frac{n}{k} \right\rceil \le k$, then $hit(K_{m,n}) \le k$.

Theorem 3.2.1 [2]:
$hit(K_{m,n}) = \min\left\{ k : \left\lceil \frac{m}{k} \right\rceil \cdot \left\lceil \frac{n}{k} \right\rceil \le k \right\}$.

3.3 Spare Subdivided Stars

Let G be a directed graph and let P be a path in G. The length of the path P is the number of its edges. Let $e = uv$ be an edge of G. Then the vertex u is a *tail* of e and the vertex v is its *head*. A *subdivision* of G is a graph resulting from a sequence of edge subdivisions applied to G. A subdivision of an edge $e = uv$ of the graph G is obtained by the addition of a new vertex w into G and replacement of uv with two new edges uw and wv.

The directed star $S_{a,1}$ is a graph with vertex set $V(S_{a,1}) = \{r, v_1, ..., v_a\}$ and edge set $E(S_{a,1}) = \{rv_j ; j = 1, ..., a\}$.

If we subdivide every edge of $S_{a,1}$ exactly $h - 1 \ge 0$ times, then we obtain the graph $S_{a,h}$. Clearly, $S_{a,h}$ consists of a directed path of length h which share one common vertex (the root of $S_{a,h}$).

In this section we determinate the exact value of $hit(S_{a,h})$ if $\min\{a, h\} \le 3$.

Lemma 3.3.1 [3]:
$hit(S_{a,1}) = hit(K_{1,a}) = \left| a^{1/2} \right|$ for any $a \le 1$.

Recall that the *union of graphs* G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edges sets E_1 and E_2 is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. We define similarly the union of n graphs G_j, $j = 1, ..., n$ and we denoted it $\bigcup_j G_j$.

Lemma 3.3.2 [3]:
Let $l \ge 2$ be an integer. Let $P^i = u_1^i e_1^i u_2^i e_2^i u_3^i$ be path of length 2, $i = 1, ..., \left\lfloor l^2 - \frac{l}{2} \right\rfloor$. Then exists an edge-irregular total $l-$coloring f of $\bigcup_i P_i$ such that $f(u_1^a) = f(u_1^b)$ for $a, b \in \left\{ 1, ..., \left\lfloor l^2 - \frac{l}{2} \right\rfloor \right\}$.

Lemma 3.3.3 [3]:

Let $k \geq 2$ be an integer. If $a \leq \left\lfloor k^2 - \frac{k}{2} \right\rfloor$, then $hit(S_{a,2}) \leq k$.

Lemma 3.3.4 [3]:
Let $k \geq 1$ be an integer. If $a \geq \left\lfloor k^2 - \frac{k}{2} \right\rfloor + 1$, then $hit(S_{a,2}) \geq k+1$.

Theorem 3.3.1 [3]:
$hit(S_{a,2}) = k$ if and only if $a \in \left((k-1)^2 - \frac{k-1}{2}, k^2 - \frac{k}{2} \right\rangle$, for any positive integer a.

Lemma 3.3.5 [3]:
Let $l \geq 2$ be an integer. Let $P^i = u_1^i e_1^i u_2^i e_2^i u_3^i e_3^i u_4^i$ be path of length 3, $i = 1,\ldots,\left\lfloor l^2 - \frac{2l}{3} \right\rfloor$. Then exists an edge-irregular total $l-$coloring f of $\bigcup_i P_i$ such that $f(u_1^a) = f(u_1^b)$ for $a,b \in \left\{1,\ldots,\left\lfloor l^2 - \frac{2l}{3} \right\rfloor\right\}$.

Lemma 3.3.6 [3]:
Let $k \geq 2$ be an integer. If $a \leq \left\lfloor k^2 - \frac{2k}{3} \right\rfloor$, then $hit(S_{a,3}) \leq k$.

Lemma 3.3.7 [3]:
Let $k \geq 1$ be an integer. If $a \geq \left\lfloor k^2 - \frac{2k}{3} \right\rfloor + 1$, then $hit(S_{a,3}) \geq k+1$.

Theorem 3.3.2 [3]:
$hit(S_{a,3}) = k$ if and only if $a \in \left((k-1)^2 - \frac{2(k-1)}{3}, k^2 - \frac{2k}{2} \right\rangle$, for any positive integer a.

Now assume that h is not bounded from above. This implies that $a \leq 3$.

Lemma 3.3.8 [3]:
Let $l \geq 2$ be an integer. Let $P^i = u_1^i e_1^i u_2^i \ldots u_{l+1}^i e_{l+1}^i u_{l+2}^i$ be path of length $l+1$, $i = 1,\ldots,l-1$. Then exists an edge-irregular total $l-$coloring f of $\bigcup_i P_i$ such that $f(u_1^a) = f(u_{l+2}^b) = l$ for $a,b \in \{1,\ldots,l-1\}$. Moreover, no two adjacent vertices have color l.

Corollary 3.3.1 [3]:
Let $l \geq 2$ be an integer. Let $P^i = u_1^i e_1^i u_2^i \ldots u_{l+1}^i e_{l+1}^i u_{l+2}^i$ be path of length $l+1$, $i = 1,\ldots,l^2 - l$. Then exists an edge-irregular total $l-$coloring f of $\bigcup_i P_i$ such that $f(u_1^a) = f(u_{l+2}^b) = l$ for $a,b \in \{1,\ldots,l-1\}$. Moreover, no two adjacent vertices have color l.

Lemma 3.3.9 [3]:

Let $l \geq 1$ be an integer. Let $P = u_1 e_1 u_2 ... u_l e_l u_{l+1}$ be path of length l. Then exists an edge-irregular total $l-$coloring of $\bigcup_i P_i$ such that all vertices of P have color l.

Corollary 3.3.2 [3]:
Let $l \geq 2$ (3) be an integer. Let P_1, P_2 (and P_3) be paths of length $\left\lfloor \frac{l}{2} \right\rfloor$ $\left(\left\lfloor \frac{l}{3} \right\rfloor\right)$. Then exists an edge-irregular total $l-$coloring of $\bigcup_i P_i$ such that all vertices have color l.

Lemma 3.3.10 [3]:
If $h \leq k^3$, then $hit(S_{1,h}) \leq k$.

Lemma 3.3.11 [3]:
If $h \geq k^3 + 1$, then $hit(S_{1,h}) \geq k + 1$.

Theorem 3.3.3 [3]:
$hit(S_{1,h}) = k$ if and only if $h \in \left((k-1)^3, k^3 \right\rangle$, for any positive integer h.

Lemma 3.3.12 [3]:
If $h \leq \left\lfloor \frac{k^3}{2} \right\rfloor$, then $hit(S_{2,h}) \leq k$.

Lemma 3.3.13 [3]:
If $h \geq \left\lfloor \frac{k^3}{2} \right\rfloor + 1$, then $hit(S_{2,h}) \geq k + 1$.

Theorem 3.3.4 [3]:
$hit(S_{2,h}) = k$ if and only if $h \in \left(\frac{(k-1)^3}{2}, \frac{k^3}{2} \right\rangle$, for any positive integer h.

Lemma 3.3.14 [3]:
If $h \leq \left\lfloor \frac{k^3}{3} \right\rfloor$, then $hit(S_{3,h}) \leq k$.

Lemma 3.3.15 [3]:
If $h \geq \left\lfloor \frac{k^3}{3} \right\rfloor + 1$, then $hit(S_{3,h}) \geq k + 1$.

Theorem 3.3.5 [3]:
$hit(S_{3,h}) = k$ if and only if $h \in \left(\frac{(k-1)^3}{3}, \frac{k^3}{3} \right\rangle$, for any positive integer h.

4. CONCLUSION

The aim was to generalize the concepts of observability numbers of vertices and edges of undirected graphs to total colourings of digraphs. We showed all known results of given parameter.

5. LITERATURE LIST

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